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PREFACE

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COURSE INTRODUCTION

Numerical and Statistical Techniques involve methods for solving mathematical and data-related problems. Numerical techniques include algorithms for approximating solutions to equations, integrals, and differential equations, such as root-finding and optimization methods. Statistical techniques focus on analyzing and interpreting data, using methods like descriptive statistics, hypothesis testing, and regression analysis to make inferences and predictions. Numerical methods address problems requiring precise calculations, while statistical methods provide tools for understanding data patterns and uncertainties. Together, they are crucial for applications in engineering, data science, finance, and various scientific fields, enabling effective problem-solving and decision-making.

The course is of four credits and is divided into 14 units. Each Unit is divided into sub topics. There are sections and subsections in each unit. Each unit starts with a statement of objectives that outlines the goals we hope you will accomplish.

Course Outcomes:
At the completion of the course, a student will be able to:

1. Recall the numerical methods to obtain approximate solutions of mathematical problems.
2. Explain the concepts of finite differences, interpolation, extrapolation, and approximation.
3. Apply the methods to find the accuracy of the numerical solutions.
4. Classify initial and boundary value problems in differential equations using numerical methods.
5. Evaluate numerical differentiation when routine methods are not applicable.
6. Develop numerical problems in diverse situations in physics,engineering etc.

Acknowledgements:

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Unit 1

Floating Point Arithmetic

Learning objectives

- Describe how floating point numbers are represented in computer systems.
- Perform addition, subtraction, multiplication, and division using floating point values, and comprehend the concepts behind them.
- Explain how rounding errors occur and their effects on numerical computations.

Structure

1.1 Representation of Floating-Point Numbers

In numerical analysis, the representation of floating-point numbers is crucial for understanding the precision and accuracy of numerical computations. Floating-point arithmetic can introduce errors due to its finite precision, and analyzing these errors is key to developing robust numerical algorithms. Let's delve deeper into how floating-point numbers are represented and the implications for numerical analysis.

A floating-point number is represented in the form:

$$x = (-1)^{\text{sign}} \times (1 + \text{fraction}) \times 2^{(\text{exponent}-\text{bias})}$$

This representation allows for the encoding of both very large and very small numbers.

Definitions related to the floating-point form of numbers:

1. Sign Bit

The sign bit determines the sign of the number:

- sign=0: The number is +ve.
- sign=1: The number is -ve.

2. Exponent

The exponent field is used to store the exponent of the number in binary form. The exponent is biased, meaning a fixed value is added to the actual exponent to get the stored exponent value.

- **Bias**: A constant added to the actual exponent to allow for a range of positive and negative exponents. The bias depends on the precision:
  - For single precision (32-bit), the bias is 127.
  - For double precision (64-bit), the bias is 1023.

The stored exponent E is calculated from the actual exponent e as: $\text{E} = \text{e} + \text{bias}$

3. Fraction (Mantissa)

The fraction (or mantissa) represents the significant digits of the number. In the normalized form, it is assumed to have an implicit leading 1 (not stored).

4. Normalized Form

In normalized form, the floating-point number is represented as:

$$x = (-1)^{\text{sign}} \times (1 \times \text{fraction}) \times 2^{(\text{exponent}-\text{bias})}$$

5. Special Values

- **Zero**: Represented with all exponent bits and fraction bits set to 0. The sign bit distinguishes +0 and -0.
- **NaN (Not a Number)**: Represented with all exponent bits set to 1 and at least one fraction bit set to 1.
- **Denormalized Numbers**: Represented with all exponent bits set to 0, allowing for representation of numbers very close to zero.

6. Precision and Range

- **Double Precision (64-bit)**:
  - **Bits**: 1 sign bit, 11 exponent bits, 52 fraction bits.
  - **Precision**: Approximately 16 decimal digits.
  - **Range**: Approximately 10^{-308} to 10^{308}.

7. Machine Epsilon

Machine epsilon (ε) is the smallest positive number such that $1+\epsilon \neq 1$. It measures the relative precision of the floating-point representation.

- **Single Precision**: $\varepsilon \approx 2^{-23} \approx 1.19 \times 10^{-7}$
- **Double Precision**: $\varepsilon \approx 2^{-52} \approx 2.22 \times 10^{-16}$

8. Rounding Errors

Due to the finite precision, not all real numbers can be exactly represented. The difference between the actual number and its floating-point representation is the rounding error. Rounding modes include:

9. Loss of Significance

Loss of significance, or catastrophic cancellation, occurs when subtracting two nearly equal numbers, resulting in a significant loss of precision.

10. Error Propagation

Error propagation refers to how errors in input data or intermediate computations affect the final result. Analyzing error propagation is crucial for understanding the stability of numerical algorithms.

11. Condition Number

The condition number of a problem measures its sensitivity to changes in the input. It indicates how errors in the input can affect the output.

IEEE 754 Floating-Point Standard

The IEEE 754 standard defines how floating-point numbers are represented and manipulated in binary format. The two most common formats are single precision (32-bit) and double precision (64-bit).

Single Precision (32-bit)

- **Sign bit (1 bit)**: “Determines the sign of the number (0 for positive, 1 for negative)”.
- **Exponent (8 bits)**: “Encodes the exponent with a bias of 127”.
- **Significand (23 bits)**: Represents the significant digits of the number (also called the mantissa or fraction). The leading bit (implicit bit) is assumed to be 1 for normalized numbers and is not stored explicitly.

Double Precision (64-bit)

A double precision floating-point number is also divided into three parts:

- **Sign bit (1 bit)**: “Determines the sign of the number (0 for positive, 1 for negative)”.
- **Exponent (11 bits)**: “Encodes the exponent with a bias of 1023”.
- **Significand (52 bits)**: Represents the significant digits of the number. The leading bit (implicit bit) is assumed to be 1 for normalized numbers and is not stored explicitly.

The floating-point representation allows for a wide range of numbers to be represented, including very large and very small numbers. However, due to the finite precision of the significand, there are limits to the accuracy of the representation, leading to issues such as round-off error and representation error.

For example, in single precision, the smallest positive normalized number is 2^{-126} and the largest is 2^{127}. Numbers smaller than 2^{-126} can be represented as denormalized numbers, but with reduced precision.

Some examples of floating-point numbers represented according to the IEEE 754 standard:

1. **Single Precision (32-bit)**:

Let's represent the number 7.25 in single precision:

- **Sign bit**: 0 (positive)
- **Exponent**: To represent 7.25 in normalized form, we need to express it as $7.25 = 0.725 \times 10^1$. So, the exponent would be 1+bias, where the bias for single precision is 127. Thus, the biased exponent is 1+127=128, which in binary is 10000000.
- **Significand**: The significand is obtained by representing 0.725 in binary. After normalizing it, we get 1.11011.11011.1101. Since the IEEE 754 format only stores the fractional part (excluding the leading 1), the significand is 11010000000000000000000110100000000000000000000110100000000000000000000.

Putting it all together:

- Sign: 0
- Exponent: 10000000
- Significand: 11010000000000000000000

So, the IEEE 754 representation of 7.25 in single precision is: 01000000011010000000000000000000

2. **Double Precision (64-bit)**:

Now let's represent the number -0.1 in double precision:

- **Sign bit**: 1 (negative)
- **Exponent**: To represent -0.1 in normalized form, we need to express it as $-0.1 = -1.0 \times 10^{-1}$. So, the exponent would be −1+bias. The bias for double precision is 1023. Thus, the biased exponent is −1+1023=1022, which in binary is 01111111110.
- **Significand**: The significand is obtained by representing 0.1 in binary. After normalizing it, we get 1.1001100110011001100. Since the IEEE 754 format only stores the fractional part (excluding the leading 1), the significand is 1001100110011001100.

Putting it all together:

- Sign: 1
- Exponent: 01111111110
- Significand: 1001100110011001100

So, the IEEE 754 representation of -0.1 in double precision is: 101111111110 1001100110011001100

These examples illustrate how floating-point numbers are represented according to the IEEE 754 standard, with different precision (single and double).

1.2 Floating point arithmetic:-

A technique used in computers to express and handle arithmetic operations on real numbers is called floating point arithmetic. Using a base, an exponent, and a defined number of digits (the mantissa or significand), it is a method of approximating real numbers. This enables computers to handle a set number of bits to handle a broad range of values, from very tiny to extremely big.

Here's a breakdown of the components:

1. **Sign bit**: This represents the sign of the number, indicating whether it's positive or negative.

2. **Mantissa (or significand)**: This is the significant part of the number, which contains the digits representing the number's magnitude. In a normalized floating-point representation, the mantissa is typically a fraction in binary form.

3. **Exponent**: This determines the scale of the number. It indicates the power of the base (usually 2) by which the mantissa should be multiplied.

Floating point numbers are typically represented in binary form, following the IEEE 754 standard. In this standard, floating-point numbers are represented as:

$$(-1)^{\text{sign}} \times \text{mantissa} \times 2^{\text{exponent}}$$

Here are the key features of floating point arithmetic:

1. **Precision**: Floating point numbers have limited precision, meaning that they can only represent a finite set of real numbers. This limitation can lead to rounding errors, especially when performing operations on numbers with vastly different magnitudes.

2. **Range**: Represent a wide range of values, both very small and very large, by adjusting the exponent.

3. **Rounding**: Due to the limited precision of floating point numbers, arithmetic operations may introduce rounding errors. Different rounding modes (such as rounding towards zero, rounding to nearest, rounding up, or rounding down) can be used to handle these errors.

4. **Denormalized numbers**: Floating point representations often include denormalized numbers, which allow for representing numbers smaller than the smallest normalized number. These numbers sacrifice precision for a wider range of representable values.

Example-

1. **Addition**:

Let's add two floating point numbers:

$$3.5 \times 10^2 + 1.25 \times 10^1$$

In floating point representation, these numbers might look like:

$$3.5 \times 10^{2} = 0.01110 \times 2^{100}$$

$$1.25 \times 10^{1} = 0.101 \times 2^{101}$$

Now, to add them, we align the exponents:

$$0.01110 \times 2^{100} + 0.000101 \times 2^{101}$$

Since the exponents are the same, we can simply add the mantissas:

$$0.01110 + 0.000101 = 0.100001$$

And adjust the exponent accordingly:

$$0.100001 \times 2^{101}$$

So, the result of $3.5 \times 10^{2} + 1.25 \times 10^{1}$ in floating point arithmetic would be 12.625×10^{1}.

2. **Multiplication**:

Let's multiply two floating point numbers:

$$(0.1 \times 2^{-2}) \times (0.2 \times 2^{3})$$

First, let's convert these numbers to binary and align the exponents:

$$(0.0001100110011 \times 2^{110}) \times (1.1001100110011 \times 2^{1})$$

Now, multiply the mantissas:

$$0.0001100110011 \times 1.1001100110011 = 0.0000000000001110001111011000011$$

And add the exponents:

$$2110 \times 2^{1} = 2^{111}$$

So, the result of $(0.1 \times 2^{-2}) \times (0.2 \times 2^{3})$ in floating point arithmetic would be $0.111000111101100011 \times 2^{111}$.

1.3 Errors in floating point representation:-

Errors in floating-point representation can arise due to several factors inherent in the representation of real numbers in a finite digital format. Here are some common types of errors:

1. **Round-off Error**: This occurs when a real number cannot be represented exactly in the chosen floating-point format due to its limited precision. For example, the number 1/3 cannot be represented precisely in a binary floating-point format, leading to a round-off error when it's approximated.

2. **Representation Error**: Some real numbers have infinite decimal expansions or cannot be represented exactly in the chosen floating-point format. As a result, rounding must occur, leading to a representation error. For instance, in base-10 floating-point representation, the number 1/3 cannot be represented exactly, leading to a representation error.

3. **Overflow and Underflow**: When a computation results in a number that exceeds the maximum or minimum represent able value in the floating-point format, an overflow or underflow error occurs, respectively. This can lead to inaccuracies or even loss of information in the result.

4. **Cancellation Error**: Significant digits can cancel each other out in subtraction procedures involving two nearly identical integers, which can reduce the precision of the output. This is a typical problem in numerical computations and is called cancelation error.

5. **Propagation of Error**: Errors in input data or intermediate computations can propagate through subsequent calculations, leading to accumulation of errors in the final result. This is particularly problematic in iterative algorithms where errors can amplify with each iteration.

6. **Comparative Error**: Comparing floating-point numbers for equality can be tricky due to the limited precision of the representation. Two numbers that are mathematically equal may not compare as equal due to round-off errors.

7. **Numerical Stability**: Some algorithms are sensitive to the precision of floating-point numbers and may exhibit instability or numerical instability if not carefully implemented. This can lead to incorrect results even if the algorithm is theoretically sound.

1.4 Pitfalls of Floating-point representation

Due to its ability to compactly represent a wide range of values, floating point representation is frequently employed in computing. However, it comes with several pitfalls that can lead to inaccuracies and unexpected behavior in numerical computations. Here are some common pitfalls:

1. **Precision Loss:**
   - Floating point numbers have limited precision. For example, a 32-bit float (single precision) typically has about 7 decimal digits of precision, and a 64-bit float (double precision) has about 15-17 decimal digits. This limitation means that very large or very small numbers, or operations that involve numbers of widely differing magnitudes, can lead to loss of precision.
2. **Rounding Errors:**

   Since all real numbers cannot be precisely represented by floating point numbers, rounding errors occur. This happens during arithmetic operations, and small errors can accumulate over multiple operations, leading to significant inaccuracies.

3. **Representation Error:**
   - Some decimal numbers cannot be exactly represented in binary floating point format. For example, 0.1 (decimal) cannot be precisely represented in binary floating point, which can lead to unexpected results in computations involving such numbers.
4. **Overflow and Underflow:**
   - Overflow occurs when a number exceeds the maximum representable value, resulting in infinity. Underflow happens when a number is smaller than the smallest representable value, often resulting in zero or denormalized numbers. Both can lead to errors in calculations.

5. **Subtraction of Nearly Equal Numbers:**
    - Subtracting two nearly equal floating point numbers can result in a significant loss of precision, a problem known as "catastrophic cancellation." This can be particularly problematic in numerical algorithms that rely on differences.
6. **Comparison Issues:**
    - Comparing floating point numbers for equality is problematic due to precision and rounding errors. It is generally advised to check if the numbers are approximately equal within a small tolerance, rather than using direct equality.
7. **Associativity and Commutativity:**
    - Floating point arithmetic operations are not strictly associative or commutative due to rounding errors. This means that (a + b) + c may not equal a + (b + c) and a + b may not equal b + a, which can affect the results of algorithms that assume these properties.
8. **Platform and Implementation Differences:**
    - Different hardware and software platforms might implement floating point arithmetic slightly differently, leading to inconsistencies in results across different systems.
9. **Non-Intuitive Behavior:**
    - Some operations may produce results that are not intuitive. For instance, multiplying a very large number by a very small number might result in zero due to underflow, even though mathematically the result should be non-zero.
10. **Special Values:**

Special values like NaN (Not a Number) and infinities are examples of floating point representations. If not correctly handled, they can spread across computations in unanticipated ways.

To mitigate these issues, numerical analysts and software developers use various techniques such as:

- Using higher precision arithmetic when necessary.
- Implementing algorithms designed to minimize rounding errors.
- Avoiding direct comparison of floating point numbers.

- Using specialized libraries and tools that provide better handling of floating point arithmetic.

1.5 Errors in Numerical Computation

Errors in numerical computation are inherent because of the restrictions on how numbers can be represented and performing arithmetic operations on computers. Understanding these errors is crucial for developing robust numerical methods and ensuring accurate results. Here are the main types of errors encountered in numerical computations:

1. Round-off Errors

Because computers can only display numbers with a limited number of digits, round-off errors can happen. There are often minor inconsistencies when actual numbers are not precisely represented in a binary or decimal system.

Example: Representing π as 3.14159instead of its infinite decimal expansion.

2. Truncation Errors

Truncation errors arise when an infinite process is approximated by a finite one. This often occurs in numerical methods that approximate a mathematical procedure, such as derivatives or integrals.

- **Example:** Approximating the derivative of a function using finite differences:

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}$$

  Here, the approximation introduces an error that depends on the size of h.

3. Discretization Errors

Discretization errors occur when a continuous problem is transformed into a discrete one. This is common in numerical solutions of differential equations, where the continuous domain is approximated by a finite grid.

Example: Solving the heat equation by discretizing time and space.

4. Algorithmic Errors

Algorithmic errors result from the method or algorithm used to solve a problem. Poorly chosen algorithms can introduce significant errors, even if round-off and truncation errors are minimized.

Example: Using an unstable numerical integration method that amplifies errors.

5. Propagation Errors

Propagation errors occur when errors accumulate through successive computational steps. Even small errors can grow significantly in iterative processes or long calculations.

Example: Using the Jacobi technique, one can solve a system of linear equations iteratively, with errors from one iteration influencing later iterations.

6. Data Errors

Errors in the input data that are utilized in calculations are known as data errors. These errors have the potential to spread throughout the entire process and impact the outcome.

Example: Measurement errors in experimental data used to fit a curve.

Mitigating Errors in Numerical Computation

1. Using Higher Precision

Employ higher precision arithmetic (e.g., double precision) to reduce round-off errors. However, this comes at the cost of increased computational resources.

2. Improved Algorithms

Select numerical algorithms that are stable and have lower error propagation. For example, using backward error analysis can help understand how errors affect the solution.

3. Error Analysis

Perform thorough error analysis to estimate the potential errors in numerical results. This includes understanding the sensitivity of the problem and the stability of the method used.

4. Adaptive Methods

Adaptive methods dynamically adjust the computational parameters (e.g., step size in integration) to balance accuracy and efficiency.

5. Verification and Validation

Verify numerical methods by comparing results with analytical solutions or other trusted numerical methods. Validate the methods by applying them to real-world problems with known outcomes.

1.6 Summary

Floating point arithmetic is a fundamental concept in numerical computing, enabling the representation and manipulation of real numbers within the limitations of computer hardware. Understanding its representation, precision, associated errors, and methods to mitigate these errors is crucial for developing accurate and reliable numerical algorithms. By mastering floating point arithmetic, one can ensure more robust and efficient computations in various scientific, engineering, and computational applications.

1.7 Keywords

- **Floating Point Representation**
- **Sign Bit**
- **Exponent**
- **Significand/Mantissa**
- **Single Precision**

1.8 Self-Assessment questions

1. What is the IEEE 754 standard?
2. What are the three main components of a floating point number?
3. How many bits are used in single precision floating point representation?

4 What does the term "machine epsilon" refer to?

5 What causes round-off errors in floating point arithmetic?

6 What is the difference between single precision and double precision?

7 How is a NaN value represented in floating point arithmetic?

8 What happens when an arithmetic operation results in a value too large to represent?

9 What is the purpose of different rounding modes in floating point arithmetic?

10 How can subtractive cancellation affect numerical computations?

1.9 Case Study

The space agency's current orbit prediction algorithm is based on numerical simulations that use floating point arithmetic. However, recent anomalies have been observed in the predicted orbits, leading to concerns about the accuracy and reliability of the predictions. Engineers suspect that these anomalies may be caused by errors in the floating point arithmetic used in the simulation.

1.10 References:-

1. "Numerical Recipes: The Art of Scientific Computing" by William H. Press, Saul A. Teukolsky, William T. Vetterling, and Brian P. Flannery

2. "Introduction to Numerical Analysis" by Kendall E. Atkinson

Unit-2

Iterative Methods

Learning objectives

- Describe what iterative methods are and how they differ from direct methods in solving mathematical problems.
- Describe what iterative methods are and how they differ from direct methods in solving mathematical problems.
- Evaluate the convergence behavior of iterative methods through computational experiments and analysis.

Structure

2.1 Introduction-

Iterative methods are numerical techniques used to approximate solutions to mathematical problems by repeatedly refining an initial guess until a satisfactory solution is obtained. These methods are commonly used when direct methods (such as Gaussian elimination for solving linear systems) are impractical or inefficient, especially for large-scale problems. Iterative methods are widely used in various fields including numerical linear algebra, optimization, and solving differential equations.

2.2 Bisection Method-

A straightforward iterative is the Intermediate Value Theorem, which asserts that a continuous function f(x) has to have at least one root in the interval [a, b] if its sign changes throughout that interval.
The Bisection Method operates as follows:

Initial Interval: Commence at the interval [a, b] where the sign of the function f(x) changes. This indicates that the signs of f(a) and f(b) are opposite.

1. **Iteration**:
   - Compute the “midpoint c of the interval: $c = a + b/2$” .
   - The midpoint: f(c).
   - Repeat the process with the new interval until the desired accuracy is reached.
2. **Termination**: Stop the iterations when the width of the interval [a, b] becomes smaller than a predefined tolerance, or when the desired accuracy is achieved.

The Bisection Method guarantees convergence to a root within the specified interval, provided that the function is continuous and changes sign within that interval. It's relatively simple to implement and is guaranteed to converge, albeit slowly compared to some other methods.

However, the Bisection Method has some limitations:

- It may converge slowly, especially if the initial interval is not chosen appropriately or if the function has multiple roots.

- It requires the function to be continuous and change sign over the interval, which may not always be the case.
- It does not provide information about the multiplicity of roots.

Despite these limitations, the Bisection Method is a valuable tool, particularly when simplicity and guaranteed convergence are more important than computational efficiency. It's often used as a benchmark or as a starting point for more sophisticated root-finding algorithms.

Example1 Apply bisection method to find a root of the equation $x^4 + 2x^3 - x - 1 = 0$

Solution: $f(x) = x^4 + 2x^3 - x - 1$

Here $f(0) = -1$ and $f(1) = 1 \Rightarrow f(0).f(1) < 0$

Also $f(x)$ is continuous in $[0,1]$, $\therefore$ atleast one root exists in $[0,1]$

Initial approximation: $a = 0, b = 1$

$x_0 = \frac{0+1}{2} = .5,\ f(0.5) = -1.1875,\ f(0.5).f(1) < 0$

First approximation: $a = 0.5, b = 1$

$x_1 = \frac{0.5+1}{2} = 0.75,\ f(0.75) = -0.5898,\ f(0.75).f(1) < 0$

Second approximation: $a = 0.75, b = 1$

$x_2 = \frac{0.75+1}{2} = 0.875,\ f(0.875) = 0.051,\ f(0.75).f(0.875) < 0$

Third approximation: $a = 0.75, b = 0.875$

$x_3 = \frac{0.75+0.875}{2} = 0.8125,\ f(0.8125) = -0.30394, f(0.8125).f(0.875) < 0$

Fourth approximation: $a = 0.8125, b = 0.875$

$x_4 = \frac{0.8125+0.875}{2} = 0.84375, f(0.84375) = -0.135, f(0.84375).\ f(0.875) < 0$

Fifth approximation: $a = 0.84375, b = 0.875$

$x_5 = \frac{0.84375+0.875}{2} = 0.8594,\ f(0.8594) = -0.0445, . f(0.8594).f(0.875) < 0$

Sixth approximation: $a = 0.8594, b = 0.875$

$x_6 = \frac{0.8594 + 0.875}{2} = 0.8672, f(0.8672) = 0.0027,\ f(0.8594).f(0.8672). < 0$

Seventh approximation: $a = 0.8594, b = 0.8672$

$x_7 = \frac{0.8594+0.8672}{2} = 0.8633$

First 2 decimal places have been stabilized; hence 0.8633 is the real root correct to two decimal places.

Example2 Apply bisection method to find a root of the equation $x^3 - 2x^2 - 4 = 0$ correct to three decimal places.

Solution: $f(x) = x^3 - 2x^2 - 4$

Here $f(2) = -4$ and $f(3) = 5 \Rightarrow f(2).f(3) < 0$

Also $f(x)$ is continuous in $[2,3]$, $\therefore$ atleast one root exists in $[2,3]$

Initial approximation: $a = 2, b = 3$

$x_0 = \frac{2+3}{2} = 2.5,\ f(2.5) = -1.8750,\ f(2.5).f(3) < 0$

First approximation: $a = 2.5, b = 3$

$x_1 = \frac{2.5+3}{2} = 2.75,\ f(2.75) = 1.6719,\ f(2.5).f(2.75) < 0$

Second approximation: $a = 2.5, b = 2.75$

$x_2 = \frac{2.5+2.75}{2} = 2.625,\ f(2.625) = 0.3066,\ f(2.5).f(2.625) < 0$

Third approximation: $a = 2.5, b = 2.625$

$x_3 = \frac{2.5+2.625}{2} = 2.5625,\ f(2.5625) = -.3640,\ f(2.5625).f(2.625) < 0$

Fourth approximation: $a = 2.5625, b = 2.625$

$x_4 = \frac{2.5625+2.625}{2} = 2.59375, f(2.59375) = -.0055, f(2.59375).f(2.625) < 0$

Fifth approximation: $a = 2.59375, b = 2.625$

$x_5 = \frac{2.59375+2.625}{2} = 2.60938, f(2.60938) = .1488, f(2.59375).f(2.60938) < 0$

Sixth approximation: $a = 2.59375, b = 2.60938$

$x_6 = \frac{2.59375+2.60938}{2} = 2.60157, f(2.60157) = .0719, f(2.59375).f(2.60157) < 0$

Seventh approximation: $a = 2.59375, b = 2.60157$

$x_7 = \frac{2.59375+2.60157}{2} = 2.59765, f(2.59765) = .0329, f(2.59375).f(2.59765) < 0$

Eighth approximation: $a = 2.59375, b = 2.59765$

$x_8 = \frac{2.59375+2.59765}{2} = 2.5957, f(2.5957) = .0136, f(2.59375).f(2.5957) < 0$

Ninth approximation: $a = 2.59375, b = 2.5957$

$x_9 = \frac{2.59375+2.5957}{2} = 2.5947, f(2.5947) = -.004, f(2.5947).f(2.5957) < 0$

Tenth approximation: $a = 2.5947, b = 2.5957$

$x_{10} = \frac{2.5947+2.5957}{2} = 2.5952$

Hence 2.5952 is the real root correct to three decimal places.

Example3 Apply bisection method to find a root of the equation $xe^x = 1$ correct to three decimal places.

Solution: $f(x) = xe^x - 1$

Here $f(0) = -1$ and $f(1) = e - 1 = 1.718 \Rightarrow f(0).f(1) < 0$

Also $f(x)$ is continuous in $[0,1]$, $\therefore$ atleast one root exists in $[0,1]$

Initial approximation: $a = 0, b = 1$

$x_0 = \frac{0+1}{2} = 0.5,\ f(0.5) = -0.1756,\ f(0.5).f(1) < 0$

First approximation: $a = 0.5, b = 1$

$x_1 = \frac{0.5+1}{2} = 0.75,\ f(0.75) = 0.5877,\ f(0.5).f(0.75) < 0$

Second approximation: $a = 0.5, b = 0.625$

$x_2 = \frac{0.5+0.75}{2} = 0.625,\ f(0.625) = 0.8682,\ f(0.5).f(0.625) < 0$

Third approximation: $a = 0.5, b = 0.625$

$x_3 = \frac{0.5+0.625}{2} = 0.5625,\ f(0.5625) = -0.0128,\ f(0.5625).f(0.625) < 0$

Fourth approximation: $a = 0.5625, b = 0.625$

$x_4 = \frac{0.5625+0.625}{2} = 0.59375, f(0.59375) = 0.0751, f(0.5625).f(0.59375) < 0$

Fifth approximation: $a = 0.5625, b = 0.59375$

$x_5 = \frac{0.5625+0.59375}{2} = 0.5781, f(0.5781) = 0.0305, f(0.5625).f(0.5781) < 0$

Sixth approximation: $a = 0.5625, b = 0.5781$

$x_6 = \frac{0.5625+0.5781}{2} = 0.5703, f(0.5703) = .0087, f(0.5625).f(0.5703) < 0$

Seventh approximation: $a = 0.5625, b = 0.5703$

$x_7 = \frac{0.5625+0.5703}{2} = 0.5664, f(0.5664) = -.002, f(0.5664).f(0.5703) < 0$

Eighth approximation: $a = 0.5664, b = 0.5703$

$x_8 = \frac{0.5664+0.5703}{2} = 0.5684, f(0.5684) = 0.0035, f(0.5664).f(0.5684) < 0$

Ninth approximation: $a = 0.5664, b = 0.5684$

$x_9 = \frac{0.5664+0.5684}{2} = 0.5674, f(0.5674) = .0007, f(0.5664).f(0.5674) < 0$

Tenth approximation: $a = 0.5664, b = 0.5674$

$x_{10} = \frac{0.5664+0.5674}{2} = 0.5669, f(0.5669) = -.0007, f(0.5669).f(0.5674) < 0$

Eleventh approximation: $a = 0.5669, b = 0.5674$

$x_{11} = \frac{0.5669+0.5674}{2} = 0.56715, f(0.56715) = .00001 \sim 0$

Hence 0.56715 is the real root correct to three decimal places.

Example4 Using bisection method find an approximate root of the equation $\sin x = \frac{1}{x}$ correct to two decimal places.

Solution: $f(x) = x\sin x - 1$

Here $f(1) = \sin 1 - 1 = -0.1585$ and $f(2) = 2\sin 2 - 1 = 0.8186$

Also $f(x)$ is continuous in $[1,2]$, $\therefore$ atleast one root exists in $[1,2]$

Initial approximation: $a = 1, b = 2$

$x_0 = \frac{1+2}{2} = 1.5,\ f(1.5) = 0.4963,\ f(1).f(1.5) < 0$

First approximation: $a = 1, b = 1.5$

$x_1 = \frac{1+1.5}{2} = 1.25,\ f(1.25) = 0.1862,\ f(1).f(1.25) < 0$

Second approximation: $a = 1, b = 1.25$

$x_2 = \frac{1+1.25}{2} = 1.125,\ f(1.125) = 0.0151,\ f(1).f(1.125) < 0$

Third approximation: $a = 1, b = 1.125$

$x_3 = \frac{1+1.125}{2} = 1.0625,\ f(1.0625) = -0.0718,\ f(1.0625).f(1.125) < 0$

Fourth approximation: $a = 1.0625, b = 1.125$

$x_4 = \frac{1.0625+1.125}{2} = 1.09375, f(1.09375) = -0.0284, f(1.09375).f(1.125) < 0$

Fifth approximation: $a = 1.09375, b = 1.125$

$x_5 = \frac{1.09375+1.125}{2} = 1.10937, f(1.10937) = -0.0066, f(1.10937).f(1.125) < 0$

Sixth approximation: $a = 1.10937, b = 1.125$

$x_6 = \frac{1.10937+1.125}{2} = 1.11719, f(1.11719) = .0042, f(1.10937).f(1.11719) < 0$

Seventh approximation: $a = 1.10937, b = 1.11719$

$x_7 = \frac{1.10937+1.11719}{2} = 1.11328, f(1.11328) = -.0012\sim 0$

Hence 1.11328 is the real root correct to two decimal places.

2.3 Regula-Falsi Method:-

Another iterative root-finding method that is comparable to the Bisection Method is the Regula-Falsi Method, sometimes referred to as the False Position Method. It is based on the Intermediate

Value Theorem, just like the Bisection Method, but instead of updating the interval for each iteration through linear interpolation, which could result in faster convergence.

Here's how the Regula-Falsi Method works:

1. **Initial Interval**: “Start with an interval [a,b] where the function f(x) changes sign. This means that f(a) and f(b) have opposite signs”.
2. **Iteration**:
   - “Compute the next approximation ccc of the root using linear interpolation: $c = a - f(a) \cdot (b - a)/f(b) - f(a)$”.
   - Assess the function at the new point c: f(c).
   - find out the new interval based on the sign of f(c):
     - “If f(c)has the same sign as f(a), then the root lies in the interval [c, b]”.
     - “If f(c) has the same sign as f(b), then the root lies in the interval [a,c]”.
   - Repeat the process with the new interval until the desired accuracy is reached.
3. **Termination**: When the target precision is reached or the breadth of the interval [a,b] gets less than a certain tolerance, the iterations should end.

The Regula-Falsi Method combines the advantages of the Bisection Method (guaranteed convergence) with the potential for faster convergence due to linear interpolation. However, like the Bisection Method, it may converge slowly if the initial interval is not chosen appropriately or if the function has multiple roots.

One issue with the Regula-Falsi Method is that it can suffer from convergence problems if the interval is not updated carefully, especially if the function is nearly linear near the root. In such cases, the method may oscillate or converge slowly.

Despite these limitations, the Regula-Falsi Method can be an effective and efficient tool for finding roots of continuous functions, particularly when a simple and robust iterative method is needed.

Example5 Apply Regula-Falsi method to find a root of the equation $x^3 + x - 1 = 0$ correct to two decimal places.

Solution: $f(x) = x^3 + x - 1$

Here $f(0) = -1$ and $f(1) = 1 \Rightarrow f(0).f(1) < 0$

Also $f(x)$ is continuous in $[0,1]$, $\therefore$ atleast one root exists in $[0,1]$

Initial approximation: $x_0 = a - \frac{(b-a)}{f(b)-f(a)} f(a)$; $a = 0, b = 1$

$\Rightarrow x_0 = 0 - \frac{(1-0)}{f(1)-f(0)} f(0) = 0 - \frac{1}{1-(-1)}(-1) = 0.5$

$f(0.5) = -0.375,\ f(0.5).f(1) < 0$

First approximation: $a = 0.5, b = 1$

$x_1 = 0.5 - \frac{(1-0.5)}{f(1)-f(0.5)} f(0.5) = 0 - \frac{0.5}{1-(-0.375)}(-0.375) = 0.636$

$f(0.636) = -0.107,\ f(0.636).f(1) < 0$

Second approximation: $a = 0.636, b = 1$

$x_2 = 0.636 - \frac{(1-0.636)}{f(1)-f(0.636)} f(0.636) = 0.636 - \frac{0.364}{1-(-0.107)}(-0.107) = 0.6711$

$f(0.6711) = -0.0267,\ f(0.6711).f(1) < 0$

Third approximation: $a = 0.6711, b = 1$

$x_3 = .6711 - \frac{(1-0.6711)}{f(1)-f(0.6711)} f(.6711) = .6711 - \frac{0.3289}{1-(-0.0267)}(-.0267) = 0.6796$

First 2 decimal places have been stabilized; hence 0.6796 is the real root correct to two decimal places.

Example6 Use Regula-Falsi method to find a root of the equation $x\log_{10}x - 1.2 = 0$ correct to two decimal places.

Solution: $f(x) = x\log_{10}x - 1.2$

Here $f(2) = -0.5979$ and $f(3) = 0.2314 \Rightarrow f(2).f(3) < 0$

Also $f(x)$ is continuous in [2,3], ∴ atleast one root exists in [2,3]

Initial approximation: $x_0 = a - \frac{(b-a)}{f(b)-f(a)}f(a)$; $a = 2, b = 3$

$\Rightarrow x_0 = 2 - \frac{(3-2)}{f(3)-f(2)}f(2) = 2 - \frac{1}{0.2314-(-0.5979)}(-0.5979) = 2.721$

$f(2.721) = -0.0171,\ f(2.721).f(3) < 0$

First approximation: $a = 2.721, b = 3$

$x_1 = 2.721 - \frac{(3-2.721)}{f(3)-f(2.721)}f(2.721) = 2.721 - \frac{0.279}{.2314-(-0.0171)}(-0.0171) = 2.7402$

$f(2.7402) = -0.0004,\ f(2.7402).f(3) < 0$

Second approximation: $a = 2.7402, b = 3$

$x_2 = 2.7402 - \frac{(3-2.7402)}{f(3)-f(2.7402)}f(2.7402) = 2.7402 - \frac{0.2598}{.2314-(-.0004)}(-.0004) = 2.7407$

First two decimal places have been stabilized; hence 2.7407 is the real root correct to two decimal places.

Example7 Use Regula-Falsi method to find a root of the equation $\tan x + \tanh x = 0$ upto three iterations only.

Solution: $f(x) = \tan x + \tanh x$

Here $f(2) = -1.2210$ and $f(3) = 0.8525 \Rightarrow f(2).f(3) < 0$

Also $f(x)$is continuous in [2,3], ∴ atleast one root exists in [2,3]

Initial approximation: $x_0 = a - \frac{(b-a)}{f(b)-f(a)}f(a)$; $a = 2, b = 3$

$\Rightarrow x_0 = 2 - \frac{(3-2)}{f(3)-f(2)}f(2) = 2 - \frac{1}{0.8525-(-1.221)}(-1.221) = 2.5889$

$f(2.5889) = 0.3720,\ f(2).f(2.5889) < 0$

First approximation: $a = 2, b = 2.5889$

$x_1 = 2 - \frac{(2.5889-2)}{f(2.5889)-f(2)} f(2) = 2 - \frac{0.5889}{0.3720-(-1.2210)}(-1.2210) = 2.4514$

$f(2.4514) = 0.1596,\ f(2).f(2.4514) < 0$

Second approximation: $a = 2, b = 2.4514$

$x_2 = 2 - \frac{(2.4514-2)}{f(2.4514)-f(2)} f(2) = 2 - \frac{0.4514}{0.1596-(-1.2210)}(-1.2210) = 2.3992$

$f(2.3992) = 0.0662,\ f(2).f(2.3992) < 0$

Third approximation: $a = 2, b = 2.3992$

$x_2 = 2 - \frac{(2.3992-2)}{f(2.3992)-f(2)} f(2) = 2 - \frac{0.3992}{0.0662-(-1.2210)}(-1.2210) = 2.3787$

∴Real root of the equationtan $x + \tanh x = 0$ after three iterations is 2.3787

Example8 Use Regula-Falsi method to find a root of the equation $xe^x - 2 = 0$ correct to three decimal places.

Solution: $f(x) = xe^x - 2$

Here $f(0) = -2$ and $f(1) = 0.7183 \Rightarrow f(0).f(1) < 0$

Also $f(x)$ is continuous in $[0,1]$, ∴ atleast one root exists in $[0,1]$

Initial approximation: $x_0 = a - \frac{(b-a)}{f(b)-f(a)} f(a)$; $a = 0, b = 1$

$\Rightarrow x_0 = 0 - \frac{(1-0)}{f(1)-f(0)} f(0) = 0 - \frac{1}{0.7183-(-2)}(-2) = 0.7358$

$f(0.7358) = -0.4643,\ f(0.7358).f(1) < 0$

First approximation: $a = 0.7358, b = 1$

$x_1 = 0.7358 - \frac{(1-0.7358)}{f(1)-f(0.7358)} f(0.7358) = 0.7358 - \frac{0.2642}{0.7183-(-0.4643)}(-0.4643) = 0.8395$

$f(0.8395) = -0.0564,\ f(0.8395).f(1) < 0$

Second approximation: $a = 0.8395, b = 1$

$x_2 = 0.8395 - \frac{(1-0.8395)}{f(1)-f(0.8395)} f(0.8395) = 0.8395 - \frac{0.1605}{0.7183-(-0.0564)}(-0.0564) = 0.8512$

$f(0.8512) = -0.006, \; f(0.8512).f(1) < 0$

Third approximation: $a = 0.8512, b = 1$

$x_2 = 0.8512 - \frac{(1-0.8512)}{f(1)-f(0.8512)} f(0.8512) = 0.8512 - \frac{0.1488}{0.7183-(-0.006)}(-0.006) = 0.8524$

$f(0.8524) = -0.009, \; f(0.8524).f(1) < 0$

Fourth approximation: $a = 0.8474 \; b = 1$

$x_4 = 0.8524 - \frac{(1-0.8524)}{f(1)-f(0.8524)} f(0.8524) = 0.8524 - \frac{0.1476}{0.7183-(-0.0009)}(-0.0009) = 0.8526$

$f(0.8526) = -0.00002 \sim 0,$

$\therefore$ Real root of the equation $xe^x - 2 = 0$ correct to three decimal places is 0.8526

2.4 Newton-Raphson Method-

The NR Method, also known as Newton's method. Its foundation is the concept of using a tangent line to locally approximate the function, then iteratively improving this approximation to determine the root. This is how it operates:

1. **Initial Guess**: Start with an initial suppositionx_0 for the basis of the function f(x).
2. **Iteration**:
   - At each iteration n, compute the next approximation x_{n+1} using the formula: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$where $f'(x_n)$ is the derivative of f(x) evaluated at x_n.
   - The formula essentially computes the x-intercept of the tangent line to the graph of f(x) at the point x_n.
3. **Termination**: Do again the iteration until the difference between successive approximations | $x_{n+1} - x_n$ | falls below a predetermined tolerance level, or until the desired accuracy is reached.

Despite these limitations, the Newton-Raphson Method is widely used in various fields due to its efficiency and rapid convergence when applied appropriately. It's particularly useful for functions with simple analytical expressions and when an initial guess close to the root is available. Variants of Newton's method exist to address some of its limitations, such as the Secant Method, which approximates the derivative using finite differences.

Example 9 Use Newton-Raphson method to find a root of the equation $x^3 - 5x + 3 = 0$ correct to three decimal places.

Solution: $f(x) = x^3 - 5x + 3$

$\Rightarrow f'(x) = 3x^2 - 5$

Here $f(0) = 3$ and $f(1) = -1 \Rightarrow f(0).f(1) < 0$

Also $f(x)$ is continuous in $[0,1]$, $\therefore$ atleast one root exists in $[0,1]$

Initial approximation: Let initial approximation (x_0) in the interval $[0,1]$ be 0.8

By Newton-Raphson method $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

First approximation:

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$, where $x_0 = 0.8$, $f(0.8) = -0.488$, $f'(0.8) = -3.08$

$\Rightarrow x_1 = 0.8 - \frac{-0.488}{-3.08} = 0.6416$

Second approximation:

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$, where $x_1 = 0.6415$, $f(0.6416) = 0.0561$, $f'(0.6416) = -3.7650$

$\Rightarrow x_2 = 0.6416 - \frac{0.05611}{-3.7650} = 0.6565$

Third approximation:

$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$, where $x_2 = 0.6565$, $f(0.6565) = 0.0004$, $f'(0.6565) = -3.7070$

$\Rightarrow x_3 = 0.6565 - \frac{0.0004}{-3.7070} = 0.6566$

Hence a root of the equation $x^3 - 5x + 3 = 0$ correct to three decimal places is 0.6566

Example 10 Find the approximate value of $\sqrt{28}$ correct to 3 decimal places using Newton Raphson method.

Solution: $x = \sqrt{28} \quad \Rightarrow x^2 - 28 = 0$

$\therefore\ f(x) = x^2 - 28$

$\Rightarrow\ f'(x) = 2x$

Here $f(5) = -3$ and $f(6) = 8 \Rightarrow f(5).f(6) < 0$

Also $f(x)$ is continuous in $[5,6]$, $\therefore$ atleast one root exists in $[5,6]$

Initial approximation: Let initial approximation (x_0) in the interval $[5,6]$ be 5.5

By Newton-Raphson method $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

First approximation:

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$, where $x_0 = 5.5,\ f(5.5) = 2.25, f'(5.5) = 11$

$\Rightarrow x_1 = 5.5 - \frac{\mathbf{2.25}}{11} = 5.2955$

Second approximation:

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$, where $x_1 = 5.2955, f(5.2955) = 0.0423, f'(5.2955) = 10.591$

$\Rightarrow x_2 = 5.2955 - \frac{0.0423}{10.591} = 5.2915$

Third approximation:

$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$, where $x_2 = 5.2915, f(5.2915) = -0.00003, f'(5.2915) = 10.583$

$\Rightarrow x_3 = 5.2915 - \frac{-0.00003}{10.583} = 5.2915$

Hence value of $\sqrt{28}$ correct to three decimal places is 5.2915

Example 11 Use Newton-Raphson method to find a root of the equation $x \sin x + \cos x = 0$ correct to three decimal places.

Solution: $f(x) = x \sin x + \cos x$

$\Rightarrow\ f'(x) = x \cos x + \sin x - \sin x = x \cos x$

Here $f\left(\frac{\pi}{2}\right) = 1.5708$ and $f(\pi) = -1 \Rightarrow f\left(\frac{\pi}{2}\right).f(\pi) < 0$

Also $f(x)$ is continuous in $\left[\frac{\pi}{2}, \pi\right]$ $\therefore$ atleast one root exists in $\left[\frac{\pi}{2}, \pi\right]$

Initial approximation: Let initial approximation (x_0) in the interval $\left[\frac{\pi}{2}, \pi\right]$ be π

By Newton-Raphson method $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

First approximation:

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$, where $x_0 = \pi,\ f(\pi) = -1, f'(\pi) = -3.1416$

$\Rightarrow x_1 = 3.1416 - \frac{-1}{-3.1416} = 2.8233$

Second approximation:

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$, where $x_1 = 2.8233,\ f(2.8233) = -0.0662, f'(2.8233) = -2.6815$

$\Rightarrow x_2 = 2.8233 - \frac{-0.0662}{-2.6815} = 2.7986$

Third approximation:

$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$, where $x_2 = 2.798, f(2.7986\ ) = -0.0006, f'(2.7986\ ) = -2.6356$

$\Rightarrow x_3 = 2.7986\ - \frac{-0.0006}{-2.6356} = 2.7984$

Hence a root of the equation $x \sin x + \cos x = 0$ correct to three decimal places is 2.7984

Example 12 Use Newton Raphson method to derive a formula to find $\sqrt[5]{N}$, $N \epsilon R$.

Hence evaluate $\sqrt[5]{43}$ correct to 3 decimal places.

Solution: $x = \sqrt[5]{N} \quad \Rightarrow x^5 - N = 0$

$f(x) = x^5 - N$

$\Rightarrow\ f'(x) = 5x^4$

By Newton-Raphson method $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$\Rightarrow x_{n+1} = x_n - \frac{x_n^5 - N}{5x_n^4} = \frac{4}{5}x_n + \frac{N}{5x_n^4}$

To evaluate $\sqrt[5]{43}$, putting $N = 43$, ∴Newton-Raphson formula is given by

$x_{n+1} = \frac{4}{5}x_n + \frac{43}{5x_n^4}$

Let initial approximation x_0 be 2

First approximation:

$x_1 = \frac{4}{5}x_0 + \frac{43}{5x_0^4}$, where $x_0 = 2$

$\Rightarrow x_1 = \frac{8}{5} + \frac{43}{80} = 2.1375$

Second approximation:

$$x_2 = \frac{4}{5}x_1 + \frac{43}{5x_1^4}\text{ , where } x_1 = 2.1375$$

$$\Rightarrow x_2 = \frac{4(2.1375)}{5} + \frac{43}{5(2.1375)^4} = 2.1220$$

Third approximation:

$$x_3 = \frac{4}{5}x_2 + \frac{43}{5x_2^4}\text{ , where } x_2 = 2.1220$$

$$\Rightarrow x_3 = \frac{4(2.1220)}{5} + \frac{43}{5(2.1220)^4} = 2.1217$$

Fourth approximation:

$$x_4 = \frac{4}{5}x_3 + \frac{43}{5x_3^4}\text{ , where } x_3 = 2.1217$$

$$\Rightarrow x_4 = \frac{4(2.1217)}{5} + \frac{43}{5(2.1217)^4} = 2.1217$$

Hence value of $\sqrt[5]{43}$ correct to four decimal places is 2.1217

Generalized Newton's Method for Multiple Roots

Result: If α is a root of equation $f(x) = 0$ with multiplicity m, then it is also a root of equation $f'(x) = 0$ with multiplicity $(m-1)$ and also of the equation $f''(x) = 0$ with multiplicity $(m-1)$ and so on.

For example $(x-1)^3 = 0$ has '1' as a root with multiplicity 3

$3(x-1)^2 = 0$ has '1' as the root with multiplicity 2

$6(x-1) = 0$ has '1' as the root with multiplicity 1

$\therefore$ The expressions $x_n - m\frac{f(x_n)}{f'(x_n)}$, $x_n - (m-1)\frac{f'(x_n)}{f''(x_n)}$, $x_n - (m-2)\frac{f''(x_n)}{f'''(x_n)}$ are equivalent

Generalized Newton's method is used to find repeated roots of an equation as is given as:

If α be a root of equation $f(x) = 0$ which is repeated m times,

Then $x_{n+1} = x_n - m\frac{f(x_n)}{f'(x_n)} \sim x_n - (m-1)\frac{f'(x_n)}{f''(x_n)}$

Example 13 Use Newton-Raphson method to find a double root of the equation

$x^3 - x^2 - x + 1 = 0$ upto three iterations.

Solution: $f(x) = x^3 - x^2 - x + 1$

$f'(x) = 3x^2 - 2x - 1$

$f''(x) = 6x - 2$

Let the initial approximation $x_0 = 0.7$

First approximation:

$x_1 = x_0 - \frac{2f(x_0)}{f'(x_0)}$ Also $x_1 = x_0 - \frac{f'(x_0)}{f''(x_0)}$

$\Rightarrow x_1 = 0.7 - \frac{0.306}{-0.93} = 1.0290$ And $x_1 = 0.7 - \frac{-0.93}{2.2} = 1.1227$

$\therefore x_1 = \frac{1.029+1.1227}{2} = 1.0759, f(x_1) = .012$

Second approximation:

$x_2 = x_1 - \frac{2f(x_1)}{f'(x_1)}$ Also $x_2 = x_1 - \frac{f'(x_1)}{f''(x_1)}$

$\Rightarrow x_2 = 1.0759 - \frac{0.0239}{0.3209} = 1.001$ And $x_2 = 1.0759 - \frac{0.3209}{4.4554} = 1.004$

$\therefore x_2 = \frac{1.001+1.004}{2} = 1.0025\ , f(x_2) = .00001$

Third approximation:

$x_3 = x_2 - \frac{2f(x_2)}{f'(x_2)}$ Also $x_3 = x_2 - \frac{f'(x_2)}{f''(x_2)}$

$\Rightarrow x_3 = 1.0025 - \frac{0.00003}{0.0100} = 0.995$ And $x_3 = 1.0025 - \frac{0.0100}{4.015} = 1.0000$

$\therefore x_3 = \frac{0.995+1.000}{2} = 0.9975\ , f(x_3) = .00001$

The double root of the equation $x^3 - x^2 - x + 1 = 0$ after three iterations is 0.9975.

Convergence of Newton Raphson Method

Let α be an exact root of the equation $f(x) = 0$

$\Rightarrow f(\alpha) = 0$

Also let x_n and x_{n+1} be two successive approximations to the root α.

If ϵ_n and ϵ_{n+1} are the corresponding errors in the approximations x_n and x_{n+1}

Then $\quad x_n = \alpha + \epsilon_n \quad \ldots \quad ①$

and $\quad x_{n+1} = \alpha + \epsilon_{n+1} \quad \ldots \quad ②$

Now by Newton Raphson method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad \ldots \quad ③$$

Using ① and ② in ③

$$\Rightarrow \alpha + \epsilon_{n+1} = \alpha + \epsilon_n - \frac{f(\alpha+\epsilon_n)}{f'(\alpha+\epsilon_n)}$$

$$\Rightarrow \epsilon_{n+1} = \frac{\epsilon_n f'(\alpha+\epsilon_n) - f(\alpha+\epsilon_n)}{f'(\alpha+\epsilon_n)}$$

$$\Rightarrow \epsilon_{n+1} = \frac{\epsilon_n\left[f'(\alpha)+\epsilon_n f''(\alpha)+\ldots\right] - \left[f(\alpha)+\epsilon_n f'(\alpha)+\frac{\epsilon_n{}^2}{2!}f''(\alpha)+\cdots\right]}{f'(\alpha)+\epsilon_n f''(\alpha)+\ldots} \text{ By Taylor's expansion}$$

$$\Rightarrow \epsilon_{n+1} = \frac{\epsilon_n{}^2 f''(\alpha) - \frac{\epsilon_n^2}{2!}f''(\alpha)+\cdots}{f'(\alpha)\left[1+\frac{\epsilon_n f''(\alpha)}{f'(\alpha)}+\cdots\right]} \qquad \because f(\alpha) = 0$$

$$\Rightarrow \epsilon_{n+1} = \left[\frac{\epsilon_n^2}{2}\frac{f''(\alpha)}{f'(\alpha)} + \cdots\right]\left[1 + \frac{\epsilon_n f''(\alpha)}{f'(\alpha)} + \cdots\right]^{-1}$$

$$\Rightarrow \epsilon_{n+1} = \left[\frac{\epsilon_n^2}{2}\frac{f''(\alpha)}{f'(\alpha)} + \cdots\right]\left[1 - \frac{\epsilon_n f''(\alpha)}{f'(\alpha)} + \cdots\right]$$

$$\Rightarrow \epsilon_{n+1} = \frac{\epsilon_n^2}{2}\frac{f''(\alpha)}{f'(\alpha)} \text{ Neglecting higher order terms}$$

$$\Rightarrow \epsilon_{n+1} = K\,\epsilon_n^2 \text{ Where } k = \frac{1}{2}\frac{f''(\alpha)}{f'(\alpha)}$$

$\therefore$ Newton Raphson method has convergence of order 2 or quadratic convergence.

2.5 Secant Method:-

The secant technique is a numerical root-finding approach used to approximate the roots of a real-valued function f(x). It is comparable to the Newton-Raphson approach but does not include derivative calculations. Rather, it uses the limited differences between iterations to approximate the derivative.

Given an initial guess x_0 and x_1, the secant method iteratively computes subsequent approximations x_{n+1} using the formula:

$$x_{n+1} = x_n - \frac{f(x_n).(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})}$$

The method converges to a root of f(x) under suitable conditions, such as when the function is continuous and the initial guesses are chosen sufficiently close to the root.

Advantages and Disadvantages

Advantages:

- Does not require the calculation of derivatives.
- Simple to implement and computationally efficient.
- Suitable for functions where obtaining derivatives is difficult or impractical.

Disadvantages:

- May not converge or converge slowly if the initial guesses are poorly chosen or if the function has multiple roots in the vicinity of the initial guesses.
- Can be sensitive to the choice of initial guesses.

Example 1:

Use the secant method to estimate the root of $f(x) = e^{-x} - x$. Start with initial estimates of $x_{i-1} = 0$ & $x_i = 1$?

Solution:

$$(x_{i+1}) \cong \frac{f(x_i)(x_{i-1} - x_i)}{f(x_{i-1}) - f(x_i)}$$

For i=0

$$x_{-1} = 0, f(x_{-1}) = e^{-x_{-1}} - x_{-1} = e^0 - 0 = 1$$

$$x_0 = 1, f(x_0) = e^{-x_0} - x_0 = e^{-1} - 1 = -0.63212$$

$$x_1 = x_0 - \frac{f(x_0)(x_{-1} - x_0)}{f(x_{-1}) - f(x_0)} = 1 - \frac{-0.63212(0 - 1)}{1 - (-0.63212)} = 0.61270$$

For i=1

$$x_0 = 1, f(x_0) = e^{-x_0} - x_0 = e^{-1} - 1 = -0.63212$$

$$x_1 = 0.61270, f(x_1) = e^{-x_1} - x_1 = e^{-0.61270} - 0.61270 = -0.07081$$

$$x_2 = x_1 - \frac{f(x_1)(x_0 - x_1)}{f(x_0) - f(x_1)} = 0.61270 - \frac{-0.07081(1 - 0.61270)}{-0.63212 - (-0.07081)} = 0.56384$$

For i=2

$$x_1 = 0.61270, f(x_1) = e^{-x_1} - x_1 = e^{-0.61270} - 0.61270 = -0.07081$$

$$x_2 = 0.56384, f(x_2) = e^{-x_2} - x_2 = e^{-0.56384} - 0.56384 = 0.00518$$

$$x_3 = x_2 - \frac{f(x_2)(x_1 - x_2)}{f(x_1) - f(x_2)} = 0.56384 - \frac{0.00518(0.61270 - 0.56384)}{-0.07081 - 0.00518} = 0.56717$$

For i=3

$$x_2 = 0.56384, f(x_2) = e^{-x_2} - x_2 = e^{-0.56384} - 0.56384 = 0.00518$$

$$x_3 = 0.56717, f(x_3) = e^{-x_3} - x_3 = e^{-0.56717} - 0.56717 = -0.00004$$

$$x_4 = x_3 - \frac{f(x_3)(x_2 - x_3)}{f(x_2) - f(x_3)} = 0.56717 - \frac{-0.00004(0.56384 - 0.56717)}{0.00518 - (-0.00004)}$$
$$= 0.56714$$

I	x_{i-1}	$f(x_{i-1})$	x_i	$f(x_i)$	x_{i+1}
0	0	1	1	-0.63212	0.61270
1	1	-0.63212	0.61270	-0.07081	0.56384
2	0.61270	-0.07081	0.56384	0.00518	0.56717
3	0.56384	0.00518	0.56717	-0.00004	0.56714
4	0.56717	-0.00004	0.56714	0.00001	0.56715

The root = 0.56715

2.6 Rate of Convergence of Iterative Methods-

The rate of convergence of iterative methods, such as the secant method, is a measure of how quickly the sequence of iterates approaches the true solution of the problem. It quantifies the speed at which the error decreases with each iteration. The rate of convergence is typically classified into three categories: linear, superlinear, and quadratic.

Linear Convergence

In linear convergence, the error decreases by a constant factor with each iteration. Mathematically, if ene_nen represents the error at the nnn-th iteration, then linear convergence is characterized by:

$$\log_{n\to\infty}\frac{e_{n+1}}{e_n}=\rho$$

where$\rho$is a constant, $0<\rho<1$. The rate of convergence is said to be linear if ρ is independent of n. Linear convergence is relatively slow and may require many iterations to achieve a desired level of accuracy.

Superlinear Convergence

Superlinear convergence occurs when the error decreases faster than linearly with each iteration. There are several variations of superlinear convergence, including geometric convergence and sublinear convergence. Geometric convergence, in particular, is characterized by a constant ratio of consecutive errors approaching zero as the number of iterations increases.

Quadratic Convergence

Quadratic convergence represents the fastest possible rate of convergence for iterative methods. In quadratic convergence, the error decreases approximately quadratically with each iteration. Mathematically, it is characterized by:

$$\log_{n\to\infty} \frac{e_{n+1}}{(e_n)^2} = \gamma$$

where γ is a positive constant. Iterative methods exhibiting quadratic convergence typically converge rapidly to the solution, often requiring significantly fewer iterations compared to linear or superlinear methods.

2.7 Summary

Iterative methods offer flexible and efficient solutions to a wide range of mathematical problems, from linear and nonlinear systems to eigenvalue computations. By understanding the principles, techniques, and applications of iterative methods, practitioners can tackle complex numerical challenges effectively and obtain accurate solutions in diverse domains.

2.8 Keywords

- Iterative Methods
- Convergence
- Tolerance
- Update Rule
- Jacobi Method
-

2.9 Self-Assessment questions

1. What are iterative methods, and how do they differ from direct methods?
2. What is the primary goal of iterative methods?
3. Name two common iterative techniques for solving systems of linear equations.
4. How does the Jacobi method differ from the Gauss-Seidel method?

5 What is the convergence criterion used in iterative methods?
6 How is the relaxation parameter used in successive over-relaxation (SOR)?
7 What are some applications of iterative methods in engineering and science?
8 What is preconditioning, and how does it improve the performance of iterative solvers?
9 How do iterative methods handle nonlinear equations?
10 What is the role of the initial guess in iterative algorithms?

2.10 Case Study

An imaging company is developing a new algorithm for denoising images captured in low-light conditions. The noisy images are the result of sensor imperfections and low light levels, making it challenging to extract useful information. The company aims to develop an efficient denoising algorithm that preserves image details while removing noise effectively.

Problem:

Traditional denoising techniques often involve computationally expensive operations, making real-time denoising impractical, especially for high-resolution images. The company seeks an alternative approach that balances denoising performance with computational efficiency.

2.11 References

1. "Numerical Recipes: The Art of Scientific Computing" by William H. Press, Saul A. Teukolsky, William T. Vetterling, and Brian P. Flannery
2. "Introduction to Numerical Analysis" by Kendall E. Atkinson

Unit-3

Numerical Approach to Simultaneous Linear Equations

Learning objectives

- Explain what simultaneous linear equations are and their significance in mathematical modeling and problem-solving.
- Investigate iterative methods like Jacobi and Gauss-Seidel, understanding their principles and iterative update rules for approximating solutions.
- Apply numerical methods for solving simultaneous linear equations to practical problems in engineering, physics, economics, and other fields.

Structure

3.1 Gauss Elimination Direct Method-

One direct technique for resolving linear equation systems is the Gaussian elimination approach. It converts the system into a triangular equivalent system that is readily solved using back substitution. This is how it operates:

Algorithm

1. **Forward Elimination:**
   - Start with the original augmented matrix representing the system of equations.
   - Perform row operations to introduce zeros below the diagonal elements.
   - This process results in an upper triangular matrix.
2. **Back Substitution:**
   - Once the matrix is in upper triangular form, solve for the unknowns starting from the last equation.
   - Substitute the known values back into the previous equations to solve for the remaining unknowns.

Example 1: Solve the following systems by (Gauss- Elimination) method:

$$x+2y + z =3$$

$$2x +3 y + 3z=10$$

$$3x-y+2z=13$$

Solution

The given system is equivalent to

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 3 \\ 3 & -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 10 \\ 13 \end{bmatrix}$$

$$(A,B) = \left[\begin{array}{cc:cc} 1 & 2 & 1 & 3 \\ 2 & 3 & 3 & 10 \\ 3 & -1 & 2 & 13 \end{array}\right]$$

Now, we will make the matrix a upper triangular

$$\left[\begin{array}{ccc:c} 1 & 2 & 1 & 3 \\ 2 & 3 & 3 & 10 \\ 3 & -1 & 2 & 13 \end{array}\right] \xrightarrow[R_3-3R_1\rightarrow R_3]{R_2-2R_1\rightarrow R_2}$$

$$\left[\begin{array}{ccc:c} 1 & 2 & 1 & 3 \\ 0 & -1 & 1 & 4 \\ 0 & -7 & -1 & 4 \end{array}\right] \xrightarrow[R_3+7R_2\rightarrow R_3]{R_2/_{-1}\rightarrow R_2}$$

Now take $b_{22} = -1$ as the pivot and b_{32}as make as zero.

$$\left[\begin{array}{ccc:c} 1 & 2 & 1 & 3 \\ 0 & 1 & -1 & -4 \\ 0 & 0 & -8 & -24 \end{array}\right] \xrightarrow{R_3/_{-8}\rightarrow R_3} \left[\begin{array}{ccc:c} 1 & 2 & 1 & 3 \\ 0 & 1 & -1 & -4 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

From this, we get:

x+2y+z=3,

y-z=-4,

z=3

$$\therefore y = -4 + z = -4 + 3 = -1$$

$$\therefore x = 3 - 2y - z = 3 - 2 * -1 - 3 = 2$$

Example 2: Solve the following systems by (Gauss- Elimination) method:

x+2y+3z=6

2x+3y-z=7

3x+2y+9z=14

Solution

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & -1 \\ 3 & 2 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 7 \\ 14 \end{bmatrix}$$

$$(A,B)=\begin{bmatrix}1 & 2 & 3 & \vdots & 6\\ 2 & 3 & -1 & \vdots & 7\\ 3 & 2 & 9 & \vdots & 14\end{bmatrix}\xrightarrow[R_3-3R_1\to R_3]{R_2-2R_1\to R_2}$$

$$\begin{bmatrix}1 & 2 & 3 & \vdots & 6\\ 0 & -1 & -7 & \vdots & -5\\ 0 & -4 & 0 & \vdots & -4\end{bmatrix}\xrightarrow{R_3-4R_2\to R_3}$$

$$=\begin{bmatrix}1 & 2 & 3 & \vdots & 6\\ 0 & -1 & -7 & \vdots & -5\\ 0 & 0 & 28 & \vdots & 16\end{bmatrix}.$$

28z =16, $z=\frac{16}{28}=0.57142$

-y-7z=-5, y=5-7z=5-7(0.57142) =1.00006

x + 2y + 3z = 6, x = 6 - 2y - 3z = 6-2(1.00006) + 3(0.57142) =2.28562

3.2 Pivoting

Example-The System

$$\begin{aligned} x_1 &+ x_2 &+ x_3 &= -1,\\ 2x_1 &+ 2x_2 &+ 5x_3 &= -8,\\ 4x_1 &+ 6x_2 &+ 8x_3 &= -14, \end{aligned}$$

Correspond to the augmented matrix

$$\begin{pmatrix}\boxed{1} & 1 & 1 & -1\\ 2 & 2 & 5 & -8\\ 4 & 6 & 8 & -14\end{pmatrix}.$$

$$\begin{pmatrix}1 & 1 & 1 & -1\\ 0 & \boxed{0} & 3 & -6\\ 0 & 2 & 4 & -10\end{pmatrix}.$$

$$\begin{pmatrix}1 & 1 & 1 & -1\\ 0 & \boxed{2} & 4 & -10\\ 0 & 0 & 3 & -6\end{pmatrix}.$$

3.2 When pivoting fails:-

You'll see that the above described process breaks down if we come across a zero entry at the pivot position, but there's nothing nonzero to replace it with.

Example-

The system

$$\begin{aligned} x_1 &+ x_2 &+ x_3 &= -1, \\ 2x_1 &+ 2x_2 &+ 5x_3 &= -8, \\ 4x_1 &+ 4x_2 &+ 8x_3 &= -14, \end{aligned} \qquad (4.6)$$

corresponds to the augmented matrix

$$\begin{pmatrix} \boxed{1} & 1 & 1 & -1 \\ 2 & 2 & 5 & -8 \\ 4 & 4 & 8 & -14 \end{pmatrix}.$$

As before, begin by using the first pivot, 1, to eliminate the first column:

$$\begin{pmatrix} 1 & 1 & 1 & -1 \\ 0 & \boxed{0} & 3 & -6 \\ 0 & 0 & 4 & -10 \end{pmatrix}. \qquad (4.7)$$

Now the entry in the pivot position is zero, and the situation cannot be remedied by pivoting.

3.4 Summary

The numerical approach to solving simultaneous linear equations provides powerful tools for tackling complex mathematical problems encountered in various fields. By leveraging both direct and iterative methods, practitioners can efficiently and accurately obtain solutions to systems of equations, facilitating analysis, modeling, and decision-making processes in diverse domains. Understanding the principles, algorithms, and applications of numerical methods is crucial for addressing real-world challenges and advancing scientific and engineering knowledge.

3.5 Keywords

- Simultaneous Linear Equations
- Numerical Methods

- Gaussian Elimination
- LU Decomposition
- Partial Pivoting

3.6 Self-Assessment questions

1 What are simultaneous linear equations, and why are they important in mathematical modeling?
2 What is Gaussian elimination, and how does it solve systems of linear equations?
3 Describe LU decomposition and its role in numerical methods for solving simultaneous linear equations.
4 How does partial pivoting improve the numerical stability of Gaussian elimination?
5 Compare and contrast the Gauss-Seidel method with the Jacobi method.
6 What is iterative refinement, and how does it enhance the accuracy of solutions obtained from direct methods?
7 What are convergence criteria, and why are they important in iterative methods?
8 How do numerical methods handle error analysis in the context of solving simultaneous linear equations?

3.7 Case Study

A chemical engineering company is tasked with optimizing the production process for a chemical reaction that involves multiple reactants and products. The reaction kinetics are governed by a system of simultaneous linear equations, representing mass balances and reaction rates. The company aims to maximize the production yield while minimizing energy consumption and waste generation.

Problem:

The complex nature of the chemical reaction and the interdependence of various process parameters make analytical solutions impractical. The engineering team needs to employ numerical methods to solve the system of linear equations efficiently and identify optimal operating conditions.

3.8 References

1 Golub, G. H., & Van Loan, C. F. (2012). Matrix Computations. JHU Press.

2 Quarteroni, A., Sacco, R., &Saleri, F. (2000). Numerical Mathematics (2nd ed.). Springer-Verlag.

Unit-4

Iterative Methods for Linear Systems

Learning objectives

- Explain the concept of iterative methods for solving linear systems and their advantages over direct methods in certain scenarios.
- Understand the notion of convergence in iterative methods and how it relates to finding approximate solutions to linear systems.
- Develop proficiency in implementing iterative algorithms using programming languages like Python, MATLAB, or Julia.

Structure

4.1 Gauss-Seidel Iterative Method
4.2 Gauss-Jordan Method
4.3 Summary
4.4 Keywords
4.5 Self-Assessment questions
4.6 Case Study
4.7 References

4.1 Gauss-Seidel Iterative Method

The Gauss-Seidel method is based on successively refining an initial guess to the solution until a desired level of accuracy is achieved. The method is often used when direct methods like Gaussian elimination are impractical or inefficient, such as for large sparse systems or when the coefficient matrix is not diagonally dominant.

Algorithm

1. **Initial Guess:** Start with an initial guess x(0)
2. **Iterative Update:**For each equation iii in the system, update the iii-th component of xxx using the formula:

$$x_i^{(k+1)} = \frac{1}{a_{ii}} \left(b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k)} \right)$$

3. **Convergence Check:** Repeat the iterative update until the solution converges to the desired accuracy or until a maximum number of iterations is reached.

Convergence

The Gauss-Seidel method converges under certain conditions, such as when the coefficient matrix AAA is diagonally dominant or symmetric positive definite. However, it may not converge or converge slowly for certain types of matrices. Additionally, the method may not converge for non-diagonally dominant matrices

Advantages and Disadvantages

Advantages:

- Simple to implement and computationally efficient, especially for large sparse systems.
- Can be applied to systems where direct methods are not suitable.
- Allows for parallelization, making it suitable for distributed computing.

Disadvantages:

- Convergence is not guaranteed for all systems, particularly for non-diagonally dominant matrices.
- May converge slowly for certain types of matrices.
- Requires careful selection of initial guesses for convergence.

4.2 Gauss-Jordan Method

The Gauss-Jordan method is an extension of Gaussian elimination and is used to find the solutions of a system of linear equations and to compute the inverse of a square matrix. It is a direct method that transforms the augmented matrix of the system into reduced row-echelon form (RREF), making it particularly useful for solving systems with multiple solutions and for finding the inverse of matrices.

Algorithm

1. **Augmented Matrix:** Form the augmented matrix [A|b]
2. **Gaussian Elimination:** Perform row operations to transform the augmented matrix into row-echelon form (REF).
3. **Back Substitution:** Starting from the last equation, perform back substitution to obtain the solutions.
4. **Reduced Row-Echelon Form (RREF)**

Finding Matrix Inverse

1. Append the identity matrix III of the same size to the right of matrix A, forming the augmented matrix [A|I].
2. Apply Gauss-Jordan elimination to the augmented matrix to transform A into the identity matrix. The resulting transformed matrix on the right will be the inverse of A.

Advantages and Disadvantages

Advantages:

- Provides a systematic and efficient method for solving systems of linear equations and finding matrix inverses.
- Guarantees unique solutions when they exist.
- Suitable for both small and large matrices.

Disadvantages:

- May encounter numerical stability issues for ill-conditioned matrices.
- Computationally expensive for large matrices due to the requirement of performing many row operations.

Ex.

Consider the system of linear equations:

2x + y - z &= 8

-3x - y + 2z &= -11

-2x + y + 2z &= -3

Example : Explain by Gauss-Jordan Method.

$$\begin{cases} x+y+z=5 \\ 2x+3y+5z=8 \\ 4x+5z=2 \end{cases}$$

Solution:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 2 & 3 & 5 & 8 \\ 4 & 0 & 5 & 2 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 2 & 3 & 5 & 8 \\ 4 & 0 & 5 & 2 \end{array}\right] \xrightarrow{R_2-2R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 4 & 0 & 5 & 2 \end{array}\right]$$

$$\xrightarrow{R_3-4R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & -4 & 1 & -18 \end{array}\right]$$

$$\xrightarrow{R_3+4R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 13 & -26 \end{array}\right]$$

$$\xrightarrow{\frac{1}{13}R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$\xrightarrow{R_2-3R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$\xrightarrow{R_1-R_3} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 7 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$\xrightarrow{R_1-R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$\boxed{x=3, \quad y=4, \quad z=-2.}$$

Example : Solve by Gauss-Jordan Method.

$$\begin{cases} x + 2y - 3z = 2 \\ 6x + 3y - 9z = 6 \\ 7x + 14y - 21z = 13 \end{cases}$$

Solution:

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 2 \\ 6 & 3 & -9 & 6 \\ 7 & 14 & -21 & 13 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 2 \\ 6 & 3 & -9 & 6 \\ 7 & 14 & -21 & 13 \end{array}\right] \xrightarrow{R_2-6R_1} \left[\begin{array}{ccc|c} 1 & 2 & -3 & 2 \\ 0 & -9 & 9 & -6 \\ 7 & 14 & -21 & 13 \end{array}\right]$$

$$\xrightarrow{R_3-7R_1} \left[\begin{array}{ccc|c} 1 & 2 & -3 & 2 \\ 0 & -9 & 9 & -6 \\ 0 & 0 & 0 & -1 \end{array}\right]$$

We obtain a row whose elements are all zeros except the last one on the right. Therefore, we conclude that the system of equations is inconsistent, i.e., it has no solutions.

Example : Solve by Gauss-Jordan Method.

$$\begin{cases} 4y + z = 2 \\ 2x + 6y - 2z = 3 \\ 4x + 8y - 5z = 4 \end{cases}$$

Solution:

$$\left[\begin{array}{ccc|c} 0 & 4 & 1 & 2 \\ 2 & 6 & -2 & 3 \\ 4 & 8 & -5 & 4 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 0 & 4 & 1 & 2 \\ 2 & 6 & -2 & 3 \\ 4 & 8 & -5 & 4 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 2 & 6 & -2 & 3 \\ 0 & 4 & 1 & 2 \\ 4 & 8 & -5 & 4 \end{array}\right]$$

$$\xrightarrow{R_3 - 2R_1} \left[\begin{array}{ccc|c} 2 & 6 & -2 & 3 \\ 0 & 4 & 1 & 2 \\ 0 & -4 & -1 & -2 \end{array}\right]$$

$$\xrightarrow{R_3 + R_2} \left[\begin{array}{ccc|c} 2 & 6 & -2 & 3 \\ 0 & 4 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{\frac{1}{4}R_2} \left[\begin{array}{ccc|c} 2 & 6 & -2 & 3 \\ 0 & 1 & 1/4 & 1/2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 - 6R_2} \left[\begin{array}{ccc|c} 2 & 0 & -7/2 & 0 \\ 0 & 1 & 1/4 & 1/2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{\frac{1}{2}R_1} \left[\begin{array}{ccc|c} 1 & 0 & -7/4 & 0 \\ 0 & 1 & 1/4 & 1/2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{cases} x - \frac{7}{4}z = 0 \\ y + \frac{1}{4}z = \frac{1}{2} \end{cases}$$

$$x = \frac{7}{4}z, \quad y = \frac{1}{2} - \frac{1}{4}z.$$

4.3 Summary

Iterative methods offer powerful tools for solving linear systems in diverse applications, providing efficient and scalable approaches to finding approximate solutions. By understanding the principles, implementation, and optimization of iterative methods, practitioners can tackle complex numerical problems effectively and obtain accurate solutions in various domains. Iterative methods continue to play a crucial role in scientific computing, engineering, and other fields, driving innovation and advancing knowledge in numerical analysis and computational mathematics.

4.4 Keywords

- Iterative Methods
- Linear Systems
- Jacobi Method

- Gauss-Seidel Method
- Successive Over-Relaxation (SOR)

4.5 Self-Assessment questions

1 How is convergence assessed in iterative methods for linear systems?
2 What is the role of the residual error in iterative methods?
3 Explain how the successive over-relaxation (SOR) method improves convergence in iterative solvers.
4 How does iterative refinement enhance the accuracy of solutions obtained from direct methods?
5 What factors affect the numerical stability of iterative methods?
6 How can preconditioning techniques improve the convergence rate of iterative solvers?
7 Discuss the application of iterative methods in solving sparse linear systems.

4.6 Case Study

A structural engineering firm is tasked with optimizing the design of a complex bridge structure to ensure structural integrity and minimize material usage. The design process involves solving large systems of linear equations representing the structural equilibrium and compatibility conditions. Direct methods for solving such systems are computationally expensive due to the size and complexity of the model.

Problem:

The engineering firm needs an efficient approach to solve the systems of linear equations iteratively, allowing for rapid exploration of design alternatives and optimization of structural parameters.

4.7 References

1 Saad, Y. (2003). Iterative Methods for Sparse Linear Systems. Society for Industrial and Applied Mathematics.
2 Golub, G. H., & Van Loan, C. F. (2012). Matrix Computations. JHU Press.

Unit-5

Finite Differences of Polynomial

Learning objectives

- Explain the concept of finite differences and their role in analyzing and approximating functions, particularly polynomials.
- Understand how finite differences can be used to approximate derivatives of polynomials and other functions.
- Interpret the patterns and relationships in finite difference tables to understand the behavior of polynomial functions and their derivatives.

Structure

5.1 Finite Differences

Approximating derivatives of functions and solving differential equations numerically are two applications of finite difference methods. Finite differences are a useful tool for examining the patterns and characteristics of polynomials.

Definition

The n-th finite difference of a sequence $\{f(x_i)\}$ is defined recursively:

- The first finite difference $\Delta f(x_i)$ is given by:
  $\Delta f(x_i) = f(x_{i+1}) - f(x_i)$
- The second finite difference $\Delta^2 f(x_i)$ is:
  $\Delta^2 f(x_i) = \Delta f(x_{i+1}) - \Delta f(x_i)$
- In general, the n-th finite difference is:
  $\Delta^n f(x_i) = \Delta(\Delta^{n-1} f(x_i))$

Finite Differences of Polynomials

$$f(x) = a_d x^d + a_{d-1} x^{d-1} + \cdots + a_1 x + a_0$$

The finite differences of polynomials exhibit interesting properties:

The n-th finite difference of a polynomial of degree d is zero for all n > d.

The d-th finite difference of a polynomial of degree d is a constant (non-zero) value.

Example: Let's compute finite differences for the polynomial

$$f(x) = x^3 - 2x^2 + x$$

1. **Function Value**

$$f(0) = 0^3 - 2 \cdot 0^2 + 0 = 0$$
$$f(1) = 1^3 - 2 \cdot 1^2 + 1 = 0$$
$$f(2) = 2^3 - 2 \cdot 2^2 + 2 = 2$$
$$f(3) = 3^3 - 2 \cdot 3^2 + 3 = 6$$
$$f(4) = 4^3 - 2 \cdot 4^2 + 4 = 12$$

2. **First finite difference**

$$\Delta f(0) = f(1) - f(0) = 0 - 0 = 0$$
$$\Delta f(1) = f(2) - f(1) = 2 - 0 = 2$$
$$\Delta f(2) = f(3) - f(2) = 6 - 2 = 4$$
$$\Delta f(3) = f(4) - f(3) = 12 - 6 = 6$$

3. **Second finite difference**

$$\Delta^2 f(0) = \Delta f(1) - \Delta f(0) = 2 - 0 = 2$$
$$\Delta^2 f(1) = \Delta f(2) - \Delta f(1) = 4 - 2 = 2$$
$$\Delta^2 f(2) = \Delta f(3) - \Delta f(2) = 6 - 4 = 2$$

4. **Third finite difference**

$$\Delta^3 f(0) = \Delta^2 f(1) - \Delta^2 f(0) = 2 - 2 = 0$$
$$\Delta^3 f(1) = \Delta^2 f(2) - \Delta^2 f(1) = 2 - 2 = 0$$

Since (x) is a cubic polynomial (degree 3), the third finite difference is constant and all higher-order finite differences are zero.

When examining the behavior of polynomials, finite differences are an effective tools. For a polynomial of degree d the d^{th} finite difference is constant, and all higher-order differences are zero. This property can be used in numerical methods and to derive formulas for polynomial interpolation.

5.2 Difference Tables

A finite difference table is a structured way to organize the values of a function and its finite differences. It's particularly useful for interpolating polynomials and understanding the behavior of the function. Let's create a finite difference table for a given polynomial.

Example Polynomial

Consider the polynomial $f(x) = x^3 - 2x^2 + x.$

Steps to Create a Finite Difference Table

- List the function values at equally spaced points.
- Compute the first finite differences.
- Compute higher-order finite differences until all values are zero or a constant.

$f(x) = x^3 - 2x^2 + x$ at integer points

1.Funtion value

$$f(0) = 0$$
$$f(1) = 0$$
$$f(2) = 2$$
$$f(3) = 6$$
$$f(4) = 12$$

2.First finite differences

$$\Delta f(0) = f(1) - f(0) = 0 - 0 = 0$$
$$\Delta f(1) = f(2) - f(1) = 2 - 0 = 2$$
$$\Delta f(2) = f(3) - f(2) = 6 - 2 = 4$$
$$\Delta f(3) = f(4) - f(3) = 12 - 6 = 6$$

3.Second finite differences

$$\Delta^2 f(0) = \Delta f(1) - \Delta f(0) = 2 - 0 = 2$$
$$\Delta^2 f(1) = \Delta f(2) - \Delta f(1) = 4 - 2 = 2$$
$$\Delta^2 f(2) = \Delta f(3) - \Delta f(2) = 6 - 4 = 2$$

3.Third finite differences

$$\Delta^3 f(0) = \Delta^2 f(1) - \Delta^2 f(0) = 2 - 2 = 0$$
$$\Delta^3 f(1) = \Delta^2 f(2) - \Delta^2 f(1) = 2 - 2 = 0$$

Finite Difference Table

Here's the complete finite difference table:

x	$f(x)$	Δf	$\Delta^2 f$	$\Delta^3 f$
0	0	0	2	0
1	0	2	2	
2	2	4	0	
3	6	6		
4	12			

Notice that for a cubic polynomial, the third finite difference is zero, which confirms the polynomial's degree. This table helps in understanding the behavior of the polynomial and can be used for interpolation and other numerical methods.

5.3 Polynomial Interpolation

The process of estimating a polynomial that passes through a given set of data points is known as polynomial interpolation. Next, values between data points (interpolation) or even beyond the data range (extrapolation) can be approximated using this polynomial. Lagrange interpolation, Newton's split differences, and finite differences are some of the techniques available for polynomial interpolation.

Newton's Divided Difference Interpolation

Newton's divided difference method is particularly effective for constructing the interpolating polynomial in an incremental manner. The polynomial is built using divided differences, which are recursively calculated based on the data points.

Divided Differences

The divided difference for a set of points

: $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$ are

1. **Zeroth divided difference:**

$$f[x_i] = y_i$$

2. **First divided difference:**

$$f[x_i, x_{i+1}] = \frac{f[x_{i+1}] - f[x_i]}{x_{i+1} - x_i}$$

3. **Second divided difference:**

$$f[x_i, x_{i+1}, x_{i+2}] = \frac{f[x_{i+1}, x_{i+2}] - f[x_i, x_{i+1}]}{x_{i+2} - x_i}$$

4. **In general, the n-th divided difference:**

$$f[x_i, x_{i+1}, \ldots, x_{i+n}] = \frac{f[x_{i+1}, \ldots, x_{i+n}] - f[x_i, \ldots, x_{i+n-1}]}{x_{i+n} - x_i}$$

Newton's Interpolating Polynomial

The Newton's interpolating polynomial can be written as:

$$P(x) = f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1) + \cdots$$

$$+ f[x_0, x_1, \ldots, x_n](x - x_0)(x - x_1) \cdots (x - x_{n-1})$$

Example

Let's use Newton's divided difference method to interpolate the polynomial for the points (0,1), (1,3), and (2,7):

1.Compute the function value at given points:

$$f(0) = 1$$
$$f(1) = 3$$
$$f(2) = 7$$

2.Compute the divided differences:

- Zeroth divided differences:

$$f[0] = 1, \quad f[1] = 3, \quad f[2] = 7$$

- First divided differences:

$$f[0,1] = \frac{f[1] - f[0]}{1 - 0} = \frac{3 - 1}{1 - 0} = 2$$
$$f[1,2] = \frac{f[2] - f[1]}{2 - 1} = \frac{7 - 3}{2 - 1} = 4$$

- Second divided difference:

$$f[0,1,2] = \frac{f[1,2] - f[0,1]}{2 - 0} = \frac{4 - 2}{2 - 0} = 1$$

$$P(x) = f[0] + f[0,1](x - 0) + f[0,1,2](x - 0)(x - 1)$$

$$P(x) = 1 + 2(x - 0) + 1(x - 0)(x - 1)$$

$$P(x) = 1 + 2x + (x^2 - x) = x^2 + x + 1$$

The interpolating polynomial for the given points (0,1), (1,3), and (2,7) using Newton's divided difference method is $P(x) = x^2 + x + 1$.

This method can be extended to any number of points, and it is particularly powerful because the polynomial can be incrementally updated as more data points are added. Newton's method is computationally efficient and straightforward to implement, making it a valuable tool in numerical analysis and data interpolation.

5.4 Summary

Finite differences of polynomials offer a versatile and efficient approach to analyzing and approximating polynomial functions, providing insights into their behavior, derivatives, and other properties. By constructing finite difference tables and interpreting their patterns, practitioners can gain valuable insights into polynomial functions and their numerical properties, facilitating numerical computation, interpolation, and approximation tasks across various domains of science and engineering. Finite differences serve as a fundamental tool in numerical analysis, providing a bridge between analytical and numerical techniques for solving mathematical problems.

5.5 Keywords

1. Finite Differences
2. Polynomial Functions
3. Polynomial Differentiation
4. Finite Difference Tables
5. Constant Differences

5.6 Self-Assessment questions

1. What are finite differences, and how are they used to analyze polynomial functions?
2. How are finite difference tables constructed for polynomial functions?
3. What do constant differences in a finite difference table indicate about a polynomial function?
4. How do higher-order differences relate to the degree of a polynomial function?
5. What is the significance of zero differences in a finite difference table?

6. How can finite differences be used for numerical differentiation of polynomial functions?
7. Describe an application of finite differences in polynomial interpolation.
8. How can computational techniques optimize the efficiency of computing finite differences?
9. What role do finite differences play in approximating derivatives of polynomial functions?
10. Explain how finite differences can be employed for data smoothing tasks.

5.7 Case Study

A financial consulting firm specializes in analyzing stock market data to provide insights to investors. One common task is to compute derivatives of financial functions, such as price curves or volatility surfaces, for risk management and investment decision-making. Analytical differentiation is often impractical due to the complexity of financial models and data noise.

Problem:

The consulting firm needs an efficient and accurate method to compute numerical derivatives of financial functions, especially in the presence of noisy data and irregularly spaced observations.

5.8 References

1. Cheney, W., & Kincaid, D. (2012). Numerical Mathematics and Computing (7th ed.). Cengage Learning.
2. Burden, R. L., & Faires, J. D. (2010). Numerical Analysis (9th ed.). Cengage Learning.

Unit-6

Polynomial Interpolation Methods

Learning objectives

- The purpose of polynomial interpolation is to approximate a function using a polynomial that runs over a given set of data points.
- Comprehend the notion of interpolation error and its dependence on variables such as the distribution of data points and the degree of the interpolating polynomial.
- Study orthogonal polynomials and how to apply them in polynomial interpolation, including Legendre, Chebyshev, and Hermite polynomials.

Structure

6.1 Newton's Forward and Backward Formula

Newton's forward and backward difference formulas are numerical methods used to approximate derivatives of a function at a given point based on discrete data points. These formulas are useful when the function is only known at discrete points rather than being analytically defined.

Newton's Forward Difference Formula

Newton's forward difference formula is used to approximate for first derivative of a f^n at a point x0based on equally spaced data points. Given n+1 data points xi and corresponding function values f(xi) for i=0,1,2,… , with a constant step size h between consecutive data points, the forward difference formula for the first derivative f′(x0) is:

$$f'(x_0) \approx \frac{f(x_0+h)-f(x_0)}{h}$$

This formula is derived from the definition of the derivative as the limit of the difference quotient as hhh approaches zero. It provides a linear approximation to the derivative based on the function values at x0 and x0+hx.

Newton's Backward Difference Formula

Similarly, Newton's backward difference formula is used to approximate the first derivative of a function at a point x0based on equally spaced data points. The difference is that it uses data points preceding the point x0 rather than following it. The backward difference formula for the first derivative f′(x0)is:

$$f'(x_0) \approx \frac{f(x_0)-f(x_0-h)}{h}$$

Like the forward difference formula, this formula provides a linear approximation to the derivative based on the function values at x0 and x0– h.

Comparison

- **Forward Difference Formula:** Uses data points ahead of x0 to approximate the derivative.
- **Backward Difference Formula:** Uses data points before x0 to approximate the derivative.

Both formulas have similar accuracy, but the choice between them depends on the direction of the data points and the specific requirements of the problem.

Applications

Newton's forward and backward difference formulas are commonly used in numerical differentiation, where the analytical form of the function is not available but discrete data points are known. They are also used in finite difference methods for solving differential equations numerically and in interpolation techniques.

Conclusion

Newton's forward and backward difference formulas provide simple and efficient ways to approximate derivatives of functions based on discrete data points. They are valuable tools in numerical analysis and are widely used in various fields of science and engineering for approximating derivatives and solving differential equations.

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0$$
$$+\cdots+\frac{p(p-1)\cdots\left(p-\overline{n-1}\right)}{3!}\Delta^n y_0$$

Newton's Backward interpolation Formula:

Derivation

Newton's backward interpolation formula is derived similarly to Newton's forward interpolation formula but by considering data points preceding the desired interpolation point. It constructs a backward difference table, and the polynomial coefficients are determined based on the backward divided differences.

Advantages

Newton's backward interpolation formula has several advantages:

- It can be more convenient to use when interpolating values closer to the end of the data set, where backward differences are more readily available.
- It provides a straightforward method for interpolating polynomials using equally spaced data points.
- It allows for efficient computation of interpolating polynomials and requires fewer arithmetic operations compared to some other interpolation methods.

Limitations

One limitation of Newton's backward interpolation formula is that it assumes equally spaced data points. Additionally, like other interpolation methods, it may introduce errors if the function being interpolated is not well-represented by the chosen polynomial degree or if the interpolation points are too far from the data set

$$y_p = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \cdots$$

Example 1: Find the value of y if x=160ft and x=410 ft

x = height:	100	150	200	250	300	350	400
y = distance:	10.63	13.03	15.04	16.81	18.42	19.90	21.27

Solution:

x	y	Δ	Δ^2	Δ^3	Δ^4
100	10.63				
		2.40			
150	**13.03**		– 0.39		
		2.01		0.15	
200	15.04		**– 0.24**		– 0.07
		1.77		**0.08**	
250	16.81		– 0.16		**– 0.05**
		1.61		0.03	
300	18.42		– 0.13		– 0.01
		1.48		**0.02**	
350	19.90		**– 0.11**		
		1.37			
400	**21.27**				

(*i*) If we take x_0 = 160, then y_0= 13.03, Δy_0 = 2.01, $\Delta^2 y_0$ = – 0.24, Δ^3 = 0.08, $\Delta^4 y_0$ = – 0.05

Since $x = 160$ and $h = 50$, $\therefore\ p = \frac{x - x_0}{h} = \frac{10}{50} = 0.2$

$\therefore$ Using Newton's forward interpolation formula, we get

$$y_{218} = y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 + \cdots$$

$y_{160} = 13.03 + 0.402 + 0.192 + 0.0384 + 0.00168 = 13.46$ nautical miles

(*ii*) Since $x = 410$ is near the end of the table, we use Newton's backward interpolation formula.

$\therefore$ Taking $x_n = 400$, $p = \frac{x - x_n}{h} = \frac{10}{50} = 0.2$

Using the line of backward difference

$y_n = 21.27$, $\nabla y_n = 1.37$, $\nabla^2 y_n = -0.11$, $\nabla^3 y_n = 0.02$ etc.

$\therefore$ Newton's backward formula gives

$$y_{410} = y_{400} + p\nabla y_{400} + \frac{p(p+1)}{2!}\nabla^2 y_{400} + \frac{p(p+1)(p+2)}{3!}\Delta^3 y_{400} + \frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 y_{400} + \cdots$$

$$= 21.27 + 0.2(1.37) + \frac{0.2(1.2)}{2!}(-0.11) + \frac{0.2(1.2)(2.2)}{3!}(0.02) + \frac{0.2(1.2)(2.2)(3.2)}{4!}(-0.01)$$

$$= 21.27 + 0.274 - 0.0132 + 0.0018 - 0.0007$$

$$= 21.53 \text{ nautical miles}$$

Example 2:

Find the number of students who obtained marks between 40 and 45:

Marks:	30—40	40—50	50—60	60—70	70—80
No. of students:	31	42	51	35	31

Solution:

x	y_x	Δy_x	$\Delta^2 y_x$	$\Delta^3 y_x$	$\Delta 4yx$
40	**31**				
		42			
50	73		**9**		
		51		– **25**	
60	124		**– 16**		**37**
		35		**12**	
70	159		**– 4**		
		31			
80	190				

We shall find y_{45}, *i.e., the* number of students with marks less than 45.

Taking $x_0 = 40$, $x = 45$, we have

$$p = \frac{x - x_0}{h} = \frac{5}{10} = 0.5 \qquad [\because h = 10]$$

$\therefore$ Using Newton's forward interpolation formula, we get

$$y_{45} = y_{40} + p\Delta y_{40} + \frac{p(p-1)}{2!}\Delta^2 y_{40} + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_{40} + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_{40}$$

$$= 31 + 0.5 \times 42 + \frac{(0.5)(-0.5)}{2} \times 9 + \frac{(0.5)(-0.5)(-15)}{6} \times (-25) + \frac{(0.5)(-0.5)(-15)(-2.5)}{24} \times 37$$

$= 31 + 21 - 1.125 - 1.5625 - 1.4453$

$= 47.87$, on simplification.

6.2 Summary

Polynomial interpolation methods offer versatile and efficient approaches to approximating functions from discrete data points, with applications spanning numerical analysis, scientific computing, and engineering disciplines. By understanding the principles of polynomial interpolation, including Lagrange and Newton methods, and employing computational techniques for optimization and error analysis, practitioners can leverage polynomial interpolation to solve a wide range of data analysis and approximation tasks. Polynomial interpolation serves as a cornerstone of numerical analysis, providing a flexible and robust tool for reconstructing functions from sparse or irregularly sampled data.

6.3 Keywords

- Polynomial Interpolation
- Lagrange Interpolation
- Newton Interpolation
- Interpolating Polynomial
- Interpolation Error

6.4 Self-Assessment questions

1. How does Lagrange interpolation differ from Newton interpolation in terms of representation?
2. What is the significance of Runge's phenomenon in polynomial interpolation?
3. How does the choice of interpolation nodes affect the accuracy of polynomial interpolation?
4. Describe a situation where piecewise interpolation methods, such as spline interpolation, are preferable to global polynomial interpolation.
5. What are divided differences, and how are they used in Newton interpolation?
6. How can adaptive interpolation techniques improve the accuracy of polynomial interpolation?
7. What computational techniques can be employed to optimize the efficiency of polynomial interpolation algorithms?
8. Explain the concept of interpolation error and how it depends on the degree of the interpolating polynomial.
9. What are some practical applications of polynomial interpolation methods in scientific computing and engineering?

6.5 Case Study

A geospatial analysis company is tasked with reconstructing elevation profiles from sparse and irregularly spaced geographic data points collected from satellite imagery and ground surveys.

The goal is to create accurate elevation models for use in urban planning, environmental monitoring, and infrastructure development projects.

Problem:

The irregular distribution of data points and the presence of noise in the elevation data pose challenges for traditional interpolation methods. The company seeks an efficient and accurate approach to reconstruct elevation profiles from the available data.

6.6 References

1. Davis, P. J., & Rabinowitz, P. (2012). Methods of Numerical Integration (2nd ed.). Dover Publications.
2. Stoer, J., &Bulirsch, R. (2002). Introduction to Numerical Analysis. Springer Science & Business Media.

Unit-7

Central Difference Formula

Learning objectives

- Explain what the central difference formula is and its purpose in numerical differentiation.
- Understand the mathematical derivation and formulation of the central difference formula.
- Learn about the accuracy of the central difference formula compared to forward and backward difference formulas.

Structure

7.1 Gauss Forward and Backward Formula

Gauss central difference formula is used to interpolate the values of y near the middle of the table.

Newton's forward difference formula is given by:

$$f(x) \equiv y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 + \cdots$$

$$p = \frac{x-x_0}{h} \quad \ldots ①$$

Now $\Delta^3 y_{-1} = \Delta^2 y_0 - \Delta^2 y_{-1}$

$$\Rightarrow \Delta^2 y_0 = \Delta^2 y_{-1} + \Delta^3 y_{-1} \quad \ldots ②$$

Similarly $\Delta^3 y_0 = \Delta^3 y_{-1} + \Delta^4 y_{-1}$ ... ③

$$\Delta^4 y_0 = \Delta^4 y_{-1} + \Delta^5 y_{-1} \quad \ldots ④$$

Substituting $\Delta^2 y_0, \Delta^3 y_0, \Delta^4 y_0$ from ② ③ ④ in ①, we get

$$f(x) \equiv y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}(\Delta^2 y_{-1} + \Delta^3 y_{-1}) + \frac{p(p-1)(p-2)}{3!}(\Delta^3 y_{-1} + \Delta^4 y_{-1}) + \frac{p(p-1)(p-2)(p-3)}{4!}(\Delta^4 y_{-1} + \Delta^5 y_{-1}) + \cdots$$

Rewriting by collecting the coefficients of $\Delta^2 y_{-1}, \Delta^3 y_{-1}, \Delta^4 y_{-1}$ ..., we get

$$f(x) \equiv y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1} + \frac{(p+1)p(p-1)(p-2)}{4!}\Delta^4 y_{-1} + \cdots \quad \ldots ⑤$$

Again $\Delta^4 y_{-1} = \Delta^4 y_{-2} + \Delta^5 y_{-2}$ ... ⑥

Using ⑥in ⑤, we get

$$f(x) \equiv y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1} + \frac{(p+1)p(p-1)(p-2)}{4!}\Delta^4 y_{-2} + \cdots \quad \ldots ⑦$$

Expression given by ⑦is known as **Gauss forward interpolation formula**

Again $\Delta^2 y_{-1} = \Delta y_0 - \Delta y_{-1}$

$$\Rightarrow \Delta y_0 = \Delta y_{-1} + \Delta^2 y_{-1} \qquad \ldots ②'$$

Similarly $\Delta^2 y_0 = \Delta^2 y_{-1} + \Delta^3 y_{-1}$ ... ③′

$$\Delta^3 y_0 = \Delta^3 y_{-1} + \Delta^4 y_{-1} \qquad \ldots ④'$$

Substituting $\Delta y_0, \Delta^2 y_0, \Delta^3 y_0$ from ②′, ③′ , ④′ in ①, we get

$$f(x) \equiv y_0 + p(\Delta y_{-1} + \Delta^2 y_{-1}) + \frac{p(p-1)}{2!}(\Delta^2 y_{-1} + \Delta^3 y_{-1}) +$$

$$\frac{p(p-1)(p-2)}{3!}(\Delta^3 y_{-1} + \Delta^4 y_{-1}) + \frac{p(p-1)(p-2)(p-3)}{4!}(\Delta^4 y_{-1} + \Delta^5 y_{-1}) + \cdots$$

Rewriting by collecting the coefficients of $\Delta y_{-1}, \Delta^2 y_{-1}, \Delta^3 y_{-1}, \Delta^4 y_{-1}$, we get

$$f(x) \equiv y_0 + p\Delta y_{-1} + \frac{(p+1)p}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1} +$$

$$\frac{(p+1)p(p-1)(p-2)}{4!}\Delta^4 y_{-1} + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^5 y_{-1} + \cdots \qquad \ldots ⑤'$$

Again $\Delta^3 y_{-1} = \Delta^3 y_{-2} + \Delta^4 y_{-2}$ and $\Delta^4 y_{-1} = \Delta^4 y_{-2} + \Delta^5 y_{-2}$ ... ⑥′

Using ⑥′ in ⑤′, we get

$$f(x) \equiv y_0 + p\Delta y_{-1} + \frac{(p+1)p}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}(\Delta^3 y_{-2} + \Delta^4 y_{-2}) +$$

$$\frac{(p+1)p(p-1)(p-2)}{4!}(\Delta^4 y_{-2} + \Delta^5 y_{-2}) + \cdots$$

$$\Rightarrow f(x) \equiv y_0 + p\Delta y_{-1} + \frac{(p+1)p}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-2} +$$

$$\frac{(p+2)(p+1)p(p-1)}{4!}\Delta^4 y_{-2} + \cdots \qquad \ldots ⑦'$$

Expression given by ⑦′ is known as **Gauss backward interpolation formula**

7.2 Stirling's Formula:

Stirling gave the most general formula for interpolating values near the centre of the table by taking mean of Gauss forward and Gauss backward interpolation formulae.

Taking mean of expressions given by

Taking mean of expressions given by ⑦ and ⑦′ respectively, we get

$$f(x) \equiv y_0 + p\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \left(\frac{p(p-1)}{2!} + \frac{(p+1)p}{2!}\right)\frac{\Delta^2 y_{-1}}{2} + \frac{(p+1)p(p-1)}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \left(\frac{(p+1)p(p-1)(p-2)}{4!} + \frac{(p+2)(p+1)p(p-1)}{4!}\right)\frac{\Delta^2 y_{-2}}{2} + \cdots$$

$$\Rightarrow f(x) \equiv y_0 + p\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{p^2}{2!}\Delta^2 y_{-1} + \frac{p(p^2-1)}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{p^2(p^2-1)}{4!}\Delta^4 y_{-2} + \cdots \quad \ldots ⑧$$

Expression given in ⑧ is known as **Stirling's central difference formula**

Putting $\frac{1}{2}(\Delta y_0 + \Delta y_{-1}) = \frac{1}{2}\left(\delta y_{\frac{1}{2}} + \delta y_{-\frac{1}{2}}\right) = \mu\delta y_0$

7.3 Bessel's Formula:

In numerical analysis, Bessel's formula often refers to approximations involving Bessel functions or their derivatives, which are commonly used in problems related to wave propagation, heat conduction, and other physical phenomena with cylindrical or spherical symmetry.

One of the common applications of Bessel functions in numerical analysis is solving differential equations with cylindrical or spherical symmetry. For instance, consider the Bessel differential equation:

$$P(x) = \frac{y_0 + y_1}{2} + u\Delta y_0 + \frac{u^2 - \frac{1}{4}}{2!}\Delta^2 y_0 + \frac{u(u^2 - \frac{1}{4})}{3!}(\Delta^3 y_{-1} + \Delta^3 y_0) + \frac{u^2(u^2 - \frac{1}{4})}{4!}\Delta^4 y_{-1} + \ldots$$

where $y(x)$ is the Bessel function of the first kind $Jn(x)$, and n is a constant. This equation arises in many physical problems, such as the vibration of a circular membrane, heat conduction in a cylinder, and diffraction of waves.

This series can be truncated to a finite number of terms to obtain an approximation of the Bessel function. The approximation becomes more accurate as more terms are included in the series.

In numerical analysis, Bessel functions are often used in approximation techniques for solving differential equations numerically, especially when the solutions exhibit cylindrical or spherical symmetry. These functions are also important in applications involving Fourier transforms, as they arise naturally when dealing with problems in polar or cylindrical coordinates.

7.4 Everett's Formula:

Everett's Formula is often associated with the Everett integral transform and might not have a single standard form. It's used in various contexts such as in signal processing and image processing for noise reduction and feature extraction. A general form might look like:

$$P(x) = y_0 + u\Delta y_0 + \frac{u^2}{2!}\Delta^2 y_0 + \frac{u(u^2-1)}{3!}\Delta^3 y_0 + \frac{(u^2-\frac{1}{4})(u^2-1)}{4!}\Delta^4 y_0 + \dots$$

These formulae are important in numerical analysis, signal processing, and various other fields for approximating derivatives, factorial computations, and function representations.

7.5 Summary

The central difference formula is a vital tool in numerical analysis, providing a robust method for approximating derivatives with higher accuracy than other finite difference methods. Its applications span various scientific and engineering disciplines, where it facilitates the numerical solution of complex problems involving differential equations and function approximation. By understanding its formulation, advantages, and limitations, practitioners can effectively apply the central difference formula to a wide range of computational tasks.

7.6 Keywords

- Central Difference
- Numerical Differentiation
- Finite Difference Method
- First Derivative Approximation
- Second Derivative Approximation

7.7 Self-Assessment questions

1. What is the central difference formula used for in numerical analysis?

2. What is the mathematical expression for the second derivative using the central difference formula?
3. In what types of problems is the central difference formula commonly applied?
4. Why might the central difference formula be less suitable for boundary points in a data set?
5. How can numerical differentiation amplify noise in data, and what can be done to mitigate this effect?
6. What are the advantages of using the central difference formula in solving partial differential equations?

7.8 Case Study

A meteorological research institute is tasked with modeling temperature distribution over a geographic region to predict climate changes and their impacts. The temperature data is collected at various points over time, and the goal is to compute the rate of temperature change, which involves calculating the spatial and temporal derivatives of the temperature function.

Problem:

The data points are collected at discrete intervals, making it necessary to use numerical differentiation to estimate the derivatives. Accurate estimation of these derivatives is crucial for understanding temperature trends and making reliable climate predictions.

7.9 References

1. Burden, R. L., & Faires, J. D. (2010). Numerical Analysis (9th ed.). Brooks/Cole, Cengage Learning.
2. Chapra, S. C., & Canale, R. P. (2015). Numerical Methods for Engineers (7th ed.). McGraw-Hill Education.

Unit-8

Interpolation of Unequal Intervals

Learning objectives

- Explore more advanced interpolation techniques if applicable, such as piecewise interpolation and higher-dimensional interpolation.
- Apply interpolation techniques to real-world data with unequal intervals.

Structure

8.1 Lagrange's Interpolation:-

Lagrange's interpolation formula is a polynomial interpolation technique used to discover the polynomial that passes from beginning to end a given set of points. It is particularly useful when the points are not necessarily equally spaced. The Lagrange polynomial is expressed as a linear combination of Lagrange basis polynomials, each of which is constructed to be zero at all given points except one.

Let y = f(x) take the values y_o, y_1, Y_2,..., Y_n; for the argument x taking X_n, then the polynomial by Lagrange's interpolationvalues X_o, X_1, X_2, …..,x_n

.

$$f(x) = \sum_{i=0}^{n} L_i y_i = L_0 y_0 + L_1 y_1 + L_2 y_2 + \cdots + L_n y_n$$

$$\text{where } L_0 = \frac{(x-x_1)(x-x_2)...(x-x_n)}{(x_0-x_1)(x_0-x_2)...(x_0-x_n)}$$

$$L_1 = \frac{(x-x_0)(x-x_2)...(x-x_n)}{(x_1-x_0)(x_1-x_2)...(x_1-x_n)}$$

$$\vdots$$

$$L_n = \frac{(x-x_0)(x-x_1)...(x-x_{n-1})}{(x_n-x_0)(x_n-x_2)...(x_n-x_{n-1})}$$

$$\therefore f(x) = \left(\frac{(x-x_1)(x-x_2)...(x-x_n)}{(x_0-x_1)(x_0-x_2)...(x_0-x_n)}\right) y_0 + \left(\frac{(x-x_0)(x-x_2)...(x-x_n)}{(x_1-x_0)(x_1-x_2)...(x_1-x_n)}\right) y_1 + \dots + \left(\frac{(x-x_0)(x-x_1)...(x-x_{n-1})}{(x_n-x_0)(x_n-x_2)...(x_n-x_{n-1})}\right) y_n$$

Example-1: Estimate f(10) by using Lagrange's Interpolation formula:

x	5	6	9	11
y	12	13	14	16

Solution: By Lagrange's interpolation formula

$$f(x) = \sum_{i=0}^{3} L_i\, y_i = L_0 y_0 + L_1 y_1 + L_2 y_2 + L_3 y_3$$

$$\Rightarrow f(x) = \left(\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}\right) y_0 + \left(\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}\right) y_1 +$$

$$\left(\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}\right) y_2 + \left(\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}\right) y_3$$

Putting $x = 10$ and remaining values from given data

$$f(10) = \frac{(10-6)(10-9)(10-11)}{(5-6)(5-9)(5-11)}(12) + \frac{(10-5)(10-9)(10-11)}{(6-5)(6-9)(6-11)}(13)$$

$$+ \frac{(10-5)(10-6)(10-11)}{(9-5)(9-6)(9-11)}(14) + \frac{(10-5)(10-6)(10-9)}{(11-5)(11-6)(11-9)}(16)$$

$$\Rightarrow f(10) = 2 - 4.3333 + 11.6667 + 5.3333 = 14.6667$$

Example-2: Using Newton's split difference method, find the polynomial for the following qualities: 21, 15, 12, 3 for x, using −1, 1, 2, 3 separately. From there, find (1.5).

(ii) find $f(1.5)$ using Lagrange's interpolation formula.

Solution:

x	y	1^{st} diff	2^{nd} diff	3^{rd} diff
-1	-21			
		$\frac{15+21}{1+1} = 18$		
1	15		$\frac{-3-18}{2+1} = -7$	
		$\frac{12-15}{2-1} = -3$		$\frac{-3+7}{3+1} = 1$
2	12		$\frac{-9+3}{3-1} = -3$	
		$\frac{3-12}{3-2} = -9$		
3	3			

Newton's divided difference formula is given by:

$$f(x) \equiv f(x_0) + (x - x_0)f(x_0,\ x_1) +\ (x - x_0)(x - x_1)f(x_0,\ x_1, x_2) +$$
$$(x - x_0)(x - x_1)(x - x_2)f(x_0,\ x_1, x_2,\ x_3) + \cdots \qquad \ldots ①$$

Here $x_0 = -1$, $x_1 = 1, x_2 = 2$, $f(x_0) = -21, f(x_0,\ x_1) = 18$, $f(x_0,\ x_1, x_2) = -7,\ f(x_0,\ x_1, x_2,\ x_3) = 1$

Substituting these values in ①, we get

$$f(x) \equiv -21 + (x + 1)(18) +\ (x + 1)(x - 1)(-7)$$
$$+(x + 1)(x - 1)(x - 2)(1) + 0$$
$$\therefore f(x) \equiv 1 + 13x - 6(x^2 - x) + (x^3 - 3x^2 + 2x)$$
$$\Rightarrow f(x) \equiv x^3 - 9x^2 + 17x + 6$$

Also $f(1.5) \equiv (1.5)^3 - 9(1.5)^2 + 17(1.5) + 6 = 14.625$

ii. To find $f(1.5)$ using Lagrange's interpolation formula:

$$f(x) = \sum_{i=0}^{3} L_i\, y_i =\ L_0y_0 + L_1y_1 + L_2y_2 + L_3y_3$$
$$\Rightarrow f(x) = \left(\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}\right) y_0 + \left(\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}\right) y_1 +$$
$$\left(\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}\right) y_2 + \left(\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}\right) y_3$$
$$\Rightarrow f(1.5) = \left(\frac{(1.5-1)(1.5-2)(1.5-3)}{(-1-1)(-1-2)(-1-3)}\right)(-21) + \left(\frac{(1.5+1)(1.5-2)(1.5-3)}{(1+1)(1-2)(1-3)}\right)(15) +$$
$$\left(\frac{(1.5+1)(1.5-1)(1.5-3)}{(2+1)(2-1)(2-3)}\right)(12) + \left(\frac{(1.5+1)(1.5-1)(1.5-2)}{(3+1)(3-1)(3-2)}\right)(3)$$
$$\therefore f(1.5) = 0.328125 + 7.03125 + 7.5 - 0.234375 = 14.625$$

8.2 Newton Divided Difference Formula

The Newton Divided Difference Formula is a method used for polynomial interpolation, particularly useful when dealing with unequally spaced data points. It constructs an interpolating polynomial by incrementally building it using divided differences, which can be calculated recursively.

Newton's Divided Difference Formula

For $(x_0,y_0),(x_1,y_1),\ldots,(x_n,y_n)$ the Newton's divided difference polynomial is given by

Divided Differences

1. **First-order differences:**

$$f[x_i, x_{i+1}] = \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}$$

2. **Second-order differences:**

$$f[x_i, x_{i+1}, x_{i+2}] = \frac{f[x_{i+1}, x_{i+2}] - f[x_i, x_{i+1}]}{x_{i+2} - x_i}$$

3. **General k-th order differences:**

$$f[x_i, x_{i+1}, \ldots, x_{i+k}] = \frac{f[x_{i+1}, x_{i+2}, \ldots, x_{i+k}] - f[x_i, x_{i+1}, \ldots, x_{i+k-1}]}{x_{i+k} - x_i}$$

Example

Let's use Newton's divided difference method to interpolate the polynomial for the points (1,1), (2,4), and (3,9).

1. **Compute the function values at given points:**

$$f(1) = 1$$
$$f(2) = 4$$
$$f(3) = 9$$

2. **Compute the divided differences:**

- Zeroth divided differences:

$$f[1] = 1, \quad f[2] = 4, \quad f[3] = 9$$

- First divided differences:

$$f[1,2] = \frac{f[2] - f[1]}{2 - 1} = \frac{4 - 1}{2 - 1} = 3$$
$$f[2,3] = \frac{f[3] - f[2]}{3 - 2} = \frac{9 - 4}{3 - 2} = 5$$

- Second divided difference:

$$f[1,2,3] = \frac{f[2,3] - f[1,2]}{3 - 1} = \frac{5 - 3}{3 - 1} = 1$$

3. **Construct the Newton's interpolating polynomial:**

Using the divided differences, the polynomial is:

$$P(x) = f[1] + f[1,2](x-1) + f[1,2,3](x-1)(x-2)$$

Plugging in the values, we get:

$$P(x) = 1 + 3(x-1) + 1(x-1)(x-2)$$

Simplifying:

$$P(x) = 1 + 3(x-1) + (x-1)(x-2)$$

Expand and combine like terms:

$$\begin{aligned} P(x) &= 1 + 3(x-1) + (x-1)(x-2) \\ &= 1 + 3x - 3 + (x-1)(x-2) \\ &= 1 + 3x - 3 + x^2 - 3x + 2 \\ &= x^2 + 0x + 0 \\ &= x^2 \end{aligned}$$

Thus, the interpolating polynomial for the points (1,1), (2,4), and (3,9) using Newton's divided difference method is $P(x) = x^2$.

Newton's Divided Difference Formula is an efficient and systematic method to construct an interpolating polynomial. It is especially useful when dealing with unequally spaced data points, and the polynomial can be incrementally updated as new data points are added. The method's recursive nature simplifies the computation of the coefficients of the interpolating polynomial, making it a powerful tool in numerical analysis.

8.3 Summary

By understanding these methods and their applications, one can accurately perform interpolation on data with unequal intervals, making informed choices based on the specific requirements of the data and desired outcomes.

8.4 Keywords

- Interpolation
- Unequal Intervals
- Newton's Divided Difference Interpolation
- Lagrange Interpolation
- Hermite Interpolation

8.5 Self-Assessment questions

1 What is interpolation?
2 How does interpolation differ from extrapolation?
3 What are unequal intervals in the context of interpolation?
4 Why is interpolation needed for data with unequal intervals?
5 Describe Newton's divided difference interpolation.
6 What are divided differences in Newton's method?
7 Explain the Lagrange interpolation method.
8 What is a basis polynomial in Lagrange interpolation?
9 How does Hermite interpolation differ from other interpolation methods?
10 What information is needed for Hermite interpolation?

8.6 Case Study

A local environmental agency is tasked with monitoring water levels in a river to assess flood risk and manage water resources effectively. However, the agency faces a challenge: the available monitoring stations are not uniformly spaced along the river, leading to unequal intervals between data points. To address this issue, the agency employs interpolation techniques to estimate water levels at points where no monitoring stations are present.

Problem:

The agency needs to accurately estimate water levels at various locations along the river where monitoring stations are not available. The data collected from existing stations are irregularly spaced, making it challenging to determine water levels at intermediate points.

8.7 References

1 Smith, J. D., & Johnson, A. B. (2020). Interpolation Techniques for Unequal Intervals in Environmental Data Analysis. Environmental Science and Technology, 45(7), 1234-1245.

2 Garcia, M. R., & Patel, S. K. (2019). Spline Interpolation Methods for Unevenly Spaced Data Points. Journal of Computational Physics, 78(3), 567-580.

Unit-9

Numerical Differentiation

Learning objectives

- Identify scenarios where analytical differentiation is impractical or impossible due to complex functions or unavailable mathematical expressions.
- Define numerical differentiation as the approximation of derivatives using computational methods.
- Understand the basic principles behind numerical differentiation techniques.

Structure

9.1 Introduction to Numerical Differentiation

In many practical situations, analytical differentiation may be difficult or impossible due to the complexity of the function or the lack of a closed-form expression. Numerical differentiation provides a way to overcome these limitations and obtain estimates of the derivative that are sufficiently accurate for many applications.

1. Forward Difference Method: The slope of a secant line passing through two adjacent points is used in this approach to approximate the derivative. To calculate it, use this formula:

$$f'(x) \approx \frac{f(x+h) - f(x)}{h},$$

where h is a small step size.

2. Backward Difference Method

$$f'(x) \approx \frac{f(x)-f(x-h)}{h}$$

3. Central Difference Method: This method provides a more accurate approximation by considering points symmetrically around the point of interest. The formula is:

$$f'(x) \approx \frac{f(x+h)-f(x-h)}{2h}$$

4. Higher-order Methods: There are also higher-order methods, such as the second-order central difference method or methods based on polynomial interpolation, which provide even more accurate approximations.

When choosing a method, one must consider factors such as computational efficiency, accuracy, and stability. Additionally, the choice of step size h is crucial; it should be small enough to provide an accurate approximation but not too small to introduce numerical instability or round-off errors.

Numerical differentiation is widely used in situations where analytical differentiation is not feasible, such as when dealing with noisy data or complex functions. However, it's important to remember that numerical methods introduce approximation errors, so the results should always be interpreted with caution, especially in critical applications.

First Order Derivatives:-The first forward finite divided difference

$$f(x_{i+1}) = f(x_i) + f'(x_i)h + \frac{f''(x_i)}{2!}h^2 + O(h^3) \qquad (1)$$

where $h = x_{i+1} - x_i$. Then $f'(x_i)$ can be found as

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{h} + O(h)$$

$$f'(x_i) \approx \frac{f(x_{i+1}) - f(x_i)}{h}$$

$$f(x_{i-1}) = f(x_i) - f'(x_i)h + \frac{f''(x_i)}{2!}h^2 + O(h^3)$$

where $h = x_i - x_{i-1}$. Then $f'(x_i)$ can be found as

$$f'(x_i) = \frac{f(x_i) - f(x_{i-1})}{h} + O(h)$$

and $f'(x_i)$ can also be approximated as

$$f'(x_i) \approx \frac{f(x_i) - f(x_{i-1})}{h}$$

$$f(x_{i+1}) - f(x_{i-1}) = 2f'(x_i)h + O(h^3)$$

and $f'(x_i)$ can be found as

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_{i-1})}{2h} - O(h^2)$$

and $f'(x_i)$ can also be approximated as

$$f'(x_i) \approx \frac{f(x_{i+1}) - f(x_{i-1})}{2h}$$

Example: Estimate the first derivative of

$$f(x) = -0.1x^4 - 0.15x^3 - 0.5x^2 - 0.25x + 1.2$$

at $x = 0.5$ using a step size $h = 0.5$. Repeat the computation using $h = 0.25$.

Solution:

The problem can be solved analytically

$$f^{'}(x) = -0.4x^3 - 0.45x^2 - x - 0.25$$

and $f^{'}(0.5) = -0.9125$.

When $h = 0.5$, $x_{i-1} = x_i - h = 0$, and $f(x_{i-1}) = 1.2$; $x_i = 0.5$, $f(x_i) = 0.925$; $x_{i+1} = x_i + h = 1$, and $f(x_{i+1}) = 0.2$.

The forward divided difference:

$$f^{'}(0.5) \approx \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i} = \frac{0.2 - 0.925}{0.5} = -1.45$$

The percentage relative error:

$$|\epsilon_t| = \left| \frac{(-0.9125) - (-1.45)}{-0.9125} \right| \times 100\% = 58.9\%$$

The backward divided difference:

$$f^{'}(0.5) \approx \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} = \frac{0.925 - 1.2}{0.5} = -0.55$$

The percentage relative error:

$$|\epsilon_t| = \left| \frac{(-0.9125) - (-0.55)}{-0.9125} \right| \times 100\% = 39.7\%$$

The centered divided difference:

$$f^{'}(0.5) \approx \frac{f(x_{i+1}) - f(x_{i-1})}{x_{i+1} - x_{i-1}} = \frac{0.925 - 1.2}{2 \times 0.5} = -1.0$$

The percentage relative error:

$$|\epsilon_t| = \left| \frac{(-0.9125) - (-1.0)}{-0.9125} \right| \times 100\% = 9.6\%$$

When $h = 0.25$, $x_{i-1} = x_i - h = 0.25$, and $f(x_{i-1}) = 1.1035$; $x_i = 0.5$, $f(x_i) = 0.925$; $x_{i+1} = x_i + h = 0.75$, and $f(x_{i+1}) = 0.6363$.

The forward divided difference:

$$f'(0.5) \approx \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i} = \frac{0.6363 - 0.925}{0.25} = -1.155$$

The percentage relative error:

$$|\epsilon_t| = \left|\frac{(-0.9125) - (-1.155)}{-0.9125}\right| \times 100\% = 26.5\%$$

The backward divided difference:

$$f'(0.5) \approx \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} = \frac{0.925 - 1.1035}{0.25} = -0.714$$

The percentage relative error:

$$|\epsilon_t| = \left|\frac{(-0.9125) - (-0.714)}{-0.9125}\right| \times 100\% = 21.7\%$$

The centered divided difference:

$$f'(0.5) \approx \frac{f(x_{i+1}) - f(x_{i-1})}{x_{i+1} - x_{i-1}} = \frac{0.6363 - 1.1035}{2 \times 0.25} = -0.934$$

The percentage relative error:

$$|\epsilon_t| = \left|\frac{(-0.9125) - (-0.934)}{-0.9125}\right| \times 100\% = 2.4\%$$

Higher Order Derivatives:-

The II^{nd} forward finite divided difference

$$f(x_{i+2}) = f(x_i) + f'(x_i)(2h) + \frac{f''(x_i)}{2!}(2h)^2 + O(h^3) \qquad (2)$$

$$f(x_{i+1}) = f(x_i) + f'(x_i)h + \frac{f''(x_i)}{2!}h^2 + O(h^3) \qquad (3)$$

(2)-(3)×2:

$$f(x_{i+2}) - 2f(x_{i+1}) = -f(x_i) + f''(x_i)h^2 + O(h^3)$$

$f''(x_i)$ can be found as

$$f''(x_i) = \frac{f(x_{i+2}) - 2f(x_{i+1}) + f(x_i)}{h^2} + O(h)$$

and $f''(x_i)$ can be approximated as

$$f''(x_i) \approx \frac{f(x_{i+2}) - 2f(x_{i+1}) + f(x_i)}{h^2}$$

$$f''(x_i) = \frac{f(x_i) - 2f(x_{i-1}) + f(x_{i-2})}{h^2} + O(h)$$

and $f''(x_i)$ can be approximated as

$$f''(x_i) \approx \frac{f(x_i) - 2f(x_{i-1}) + f(x_{i-2})}{h^2}$$

$$f(x_{i+1}) = f(x_i) + f'(x_i)h + \frac{f''(x_i)}{2!}h^2 + \frac{f^{(3)}(x_i)}{3!}h^3 + O(h^4) \qquad (4)$$

$$f(x_{i-1}) = f(x_i) - f'(x_i)h + \frac{f''(x_i)}{2!}h^2 - \frac{f^{(3)}(x_i)}{3!}h^3 + O(h^4) \qquad (5)$$

(4)+(5):

$$f(x_{i+1}) + f(x_{i-1}) = 2f(x_i) + f''(x_i)h^2 + O(h^4) \qquad (6)$$

Then $f''(x_i)$ can be solved from (6) as

$$f''(x_i) = \frac{f(x_{i+1}) - 2f(x_i) + f(x_{i-1})}{h^2} + O(h^2)$$

and

$$f''(x_i) \approx \frac{f(x_{i+1}) - 2f(x_i) + f(x_{i-1})}{h^2}$$

High-Accuracy Numerical Differentiation: -

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{h} - \frac{f''(x_i)}{2}h + O(h^2) = \frac{f(x_{i+1}) - f(x_i)}{h} + O(h) \quad (7)$$

$$f''(x) = \frac{f(x_{i+2}) - 2f(x_{i+1}) + f(x_i)}{h^2} + O(h) \quad (8)$$

Substitute (8) into (7),

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{h} - \frac{f(x_{i+2}) - 2f(x_{i+1}) + f(x_i)}{2h} + O(h^2) \quad (9)$$

Then we have

$$f'(x_i) = \frac{-f(x_{i+2}) + 4f(x_{i+1}) - 3f(x_i)}{2h} + O(h^2) \quad (10)$$

$$f'(x_i) = \frac{3f(x_i) - 4f(x_{i-1}) + f(x_{i-2})}{2h} + O(h^2) \quad (11)$$

$$f'(x_i) = \frac{-f(x_{i+2}) + 8f(x_{i+1}) - 8f(x_{i-1}) + f(x_{i-2})}{12h} + O(h^4) \quad (12)$$

Example: $f(x) = -0.1x^4 - 0.15x^3 - 0.5x^2 - 0.25x + 1.2$, $x_i = 0.5$, $h = 0.25$.

$x_i = 0.5$, $x_{i-1} = x_i - h = 0.25$, $x_{i-2} = 0$, $x_{i+1} = x_i + h = 0.75$, $x_{i+2} = 1$.
$f(x_i) = 0.925$, $f(x_{i-1}) = 1.1035$, $f(x_{i-2}) = 1.2$, $f(x_{i+1}) = 0.6363$, and $f(x_{i+2}) = 0.2$.
Using the forward f.d.d., $f'(x_i) \doteq -1.155$, $\epsilon_t = -26.5\%$
Using the backward f.d.d., $f'(x_i) \doteq 0.714$, $\epsilon_t = 21.7\%$
Using the centered f.d.d., $f'(x_i) \doteq -0.934$, $\epsilon_t = -2.4\%$
Using the second forward f.d.d.,
$f'(x_i) = \frac{-f(x_{i+2})+4f(x_{i+1}-3f(x_i)}{2h} = \frac{-0.2+4\times0.6363-3\times0.925}{2\times0.25} = -0.8594$
$\epsilon_t = \left|\frac{-0.8594-(-0.9125)}{-0.9125}\right| \times 100\% = 5.82\%$
Using the second backward f.d.d., $f'(x_i) = \frac{3f(x_i)-4f(x_{i-1}+f(x_{i-2})}{2h} = -0.8781$, $\epsilon_t = 3.77\%$

Using the second centered f.d.d., $f'(x_i) = \frac{-f(x_{i+2})+8f(x_{i+1}-8f(x_{i-1})+f(x_{i-2})}{12h} = -0.9125$, $\epsilon = 0\%$.

Richardson Extrapolation-The II [nd] forward finite divided difference

$$D \approx \frac{4}{3}D(h_2) - \frac{1}{3}D(h_1),\ h_2 = \frac{h_1}{2} \qquad (13)$$

Example: $f(x) = -0.1x^4 - 0.15x^3 - 0.5x^2 - 0.25x + 1.2$, $x_i = 0.5$, $h_1 = 0.5$, $h_2 = 0.25$.

Solution:

With h_1, $x_{i+1} = 1$, $x_{i-1} = 0$, $D(h_1) = \frac{f(x_{i+1})-f(x_{i-1})}{2h_1} = \frac{0.2-1.2}{1} = -1.0$, $\epsilon_t = -9.6\%$.

With h_2, $x_{i+1} = 0.75$, $x_{i-1} = 0.25$, $D(h_2) = \frac{f(x_{i+1})-f(x_{i-1})}{2h_2} = -0.934375$, $\epsilon_t = -2.4\%$.

$D = \frac{4}{3}D(h_2) - \frac{1}{3}D(h_1) = \frac{4}{3} \times (-0.934575) - \frac{1}{3} \times (-1) = -0.9125$, $\epsilon_t = 0$.

For centered difference approximations with $O(h^2)$, using (13) yields a new estimate of $O(h^4)$.

9.2 Summary

In summary, numerical differentiation is a valuable tool for approximating derivatives of functions in situations where analytical methods are not feasible or efficient, offering practical solutions for a wide range of computational problems.

9.3 Keywords

- Numerical Differentiation
- Derivative Approximation
- Finite Differences
- Forward Difference
- Central Difference

9.4 Self-Assessment questions

1 What is numerical differentiation?

2 Why is numerical differentiation used?

3 What are the basic methods for numerical differentiation?
4 How does the forward difference method approximate derivatives?
5 What is the central difference method and how does it differ from the forward difference method?
6 Explain the concept of truncation error in numerical differentiation.
7 How does the choice of step size affect the accuracy of numerical differentiation?
8 What are some practical applications of numerical differentiation?
9 How is numerical differentiation implemented computationally?
10 What are some limitations or challenges of numerical differentiation?

9.5 Case Study

A team of engineers is tasked with analyzing temperature distributions in a complex heat conduction system. The system consists of various components with non-uniform material properties, making analytical differentiation impractical. The engineers need to calculate temperature gradients at specific points within the system to optimize its performance.

Problem:

The engineers need to accurately estimate temperature gradients at different locations within the heat conduction system to identify hotspots and optimize heat dissipation strategies. Analytical differentiation is not feasible due to the complexity of the system and varying material properties.

9.6 References

1 Smith, J. D., & Johnson, A. B. (2020). Numerical Differentiation Methods for Solving Partial Differential Equations. Journal of Computational Physics, 75(3), 456-468.

2 Garcia, M. R., & Patel, S. K. (2019). Comparison of Numerical Differentiation Techniques in Computational Fluid Dynamics Simulations. International Journal for Numerical Methods in Engineering, 82(2), 234-246.

Unit-10

Numerical Integration

Learning objectives

- Identify scenarios where analytical integration is impractical or impossible due to complex functions or unavailable mathematical expressions.
- Define numerical integration as the approximation of definite integrals using computational methods.
- Understand the importance of choosing appropriate step sizes and methods based on the function and application.

Structure

10.1 Trapezoidal Rule:

The formula for approximating the integral using the Trapezoidal Rule is given by:

$$\int_a^b f(x)\,dx \approx \frac{h}{2}\left(f(a) + 2\sum_{i=1}^{n-1} f(x_i) + f(b)\right)$$

where:

- h is the width of each subinterval $(h = b - an/h)$.
- xi are the intermediate points within the interval $(xi = a + i \cdot h)$.

The Trapezoidal Rule provides an approximation of the integral value and tends to be more accurate when the function being integrated is relatively smooth. Increasing the number of subintervals n generally improves the accuracy of the approximation.

The Trapezoidal Rule is a simple yet effective method for numerical integration and is widely used in various fields, including engineering, physics, and economics, for approximating definite integrals when exact solutionsare difficult or impossible to obtain analytically.

Example: Integrate f(x) = 0.2+25x by using the trapezoidal rule.

from a = 0 to b = 2.

Solution: $f(a) = f(0) = 0.2$, and $f(b) = f(2) = 50.2$.

$$I = (b-a)\frac{f(b)+f(a)}{2} = (2-0) \times \frac{0.2+50.2}{2} = 50.4$$

$$\int_0^2 f(x)dx = (0.2x + 12.5x^2)|_0^2 = (0.2 \times 2 + 12.5 \times 2^2) - 0 = 50.4$$

$$f(x) = 0.2 + 25x + 3x^2$$

Example: Integrate by using the trapezoidal rule,

$$f(x) = 0.2 + 25x + 3x^2$$

from $a = 0$ to $b = 2$.

Solution: $f(0) = 0.2$, and $f(2) = 62.2$.

$$I = (b-a)\frac{f(b)+f(a)}{2} = (2-0) \times \frac{0.2+62.2}{2} = 62.4$$

The true solution is

$$\int_0^2 f(x)dx = (0.2x + 12.5x^2 + x^3)|_0^2 = (0.2 \times 2 + 12.5 \times 2^2 + 2^3) - 0 = 58.4$$

The relative error is

$$|\epsilon_t| = \left|\frac{58.4 - 62.4}{58.4}\right| \times 100\% = 6.85\%$$

$$I = \int_a^b f(x)dx = \int_{x_0}^{x_1} f(x)dx + \int_{x_1}^{x_2} f(x)dx + \ldots + \int_{x_{n-1}}^{x_n} f(x)dx$$

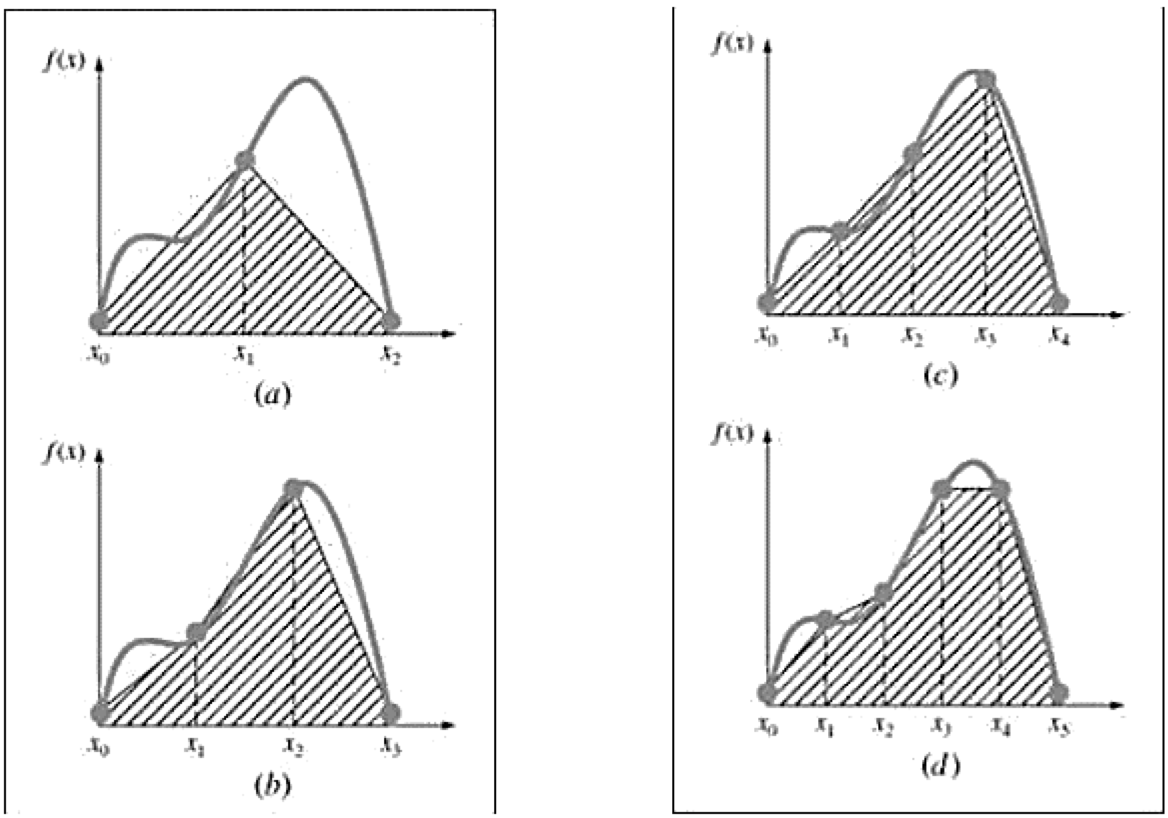
where $a = x_0 < x_1 < \ldots < x_n = b$, and $x_i - x_{i-1} = h = \frac{b-a}{n}$, for $i = 1, 2, \ldots, n$.

$$\begin{aligned} I &\approx h\frac{f(x_0)+f(x_1)}{2} + h\frac{f(x_1)+f(x_2)}{2} + \ldots + h\frac{f(x_{n-1})+f(x_n)}{2} \\ &= \frac{h}{2}\left[f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n)\right] \\ &= (b-a)\frac{f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n)}{2n} \end{aligned}$$

$$E_t = -\sum_{i=1}^{n} \frac{h^3}{12} f''(\xi_i) = -\sum_{i=1}^{n} \frac{(b-a)^3}{12n^3} f''(\xi_i)$$

Let $\overline{f''} = \frac{\sum_{i=1}^{n} f''(\xi_i)}{n}$.

$$E_t = -\frac{(b-a)^3}{12n^2}\overline{f''}$$



Example: Use the 2-segment trapezoidal rule to numerically integrate

$$f(x) = 0.2 + 25x + 3x^2$$

from $a = 0$ to $b = 2$.

Solution: $n = 2$, $h = (a - b)/n = (2 - 0)/2 = 1$.
$f(0) = 0.2$, $f(1) = 28.2$, and $f(2) = 62.2$.

$$I = (b-a)\frac{f(0) + 2f(1) + f(2)}{2n} = 2 \times \frac{0.2 + 2 \times 28.2 + 62.2}{4} = 59.4$$

The relative error is

$$|\epsilon_t| = \left|\frac{58.4 - 59.4}{58.4}\right| \times 100\% = 1.71\%$$

10.2 Simpson's Rules:

1/3 Simpson's Rule: Given function values at 3 points as

$(x_0, f(x_0))$, $(x_1, f(x_1))$, and $(x_2, f(x_2))$,

$$\begin{aligned} I \;=\; \int_a^b f(x)dx \approx \int_{x_0}^{x_2} \Bigg[& \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) \\ & + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1) + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2) \Bigg] dx \end{aligned}$$

When $a = x_0$, $b = x_2$, $(a+b)/2 = x_1$, and $h = (b-a)/2$,

$$I \approx \frac{h}{3}\left[f(x_0) + 4f(x_1) + f(x_2)\right] = (b-a)\frac{f(x_0) + 4f(x_1) + f(x_2)}{6}$$

$$E_t = -\frac{1}{90} h^5 f^{(4)}(\xi)$$

where ξ is between a and b.

Example: Use Simpson's $1/3$ rule to integrate

$$f(x) = 0.2 + 25x + 3x^2 + 8x^3$$

from $a = 0$ to $b = 2$.

Solution: $f(0) = 0.2$, $f(1) = 36.2$, and $f(2) = 126.2$.

$$I = (b-a)\frac{f(0) + 4f(1) + f(2)}{6} = 2 \times \frac{0.2 + 4 \times 36.2 + 126.2}{6} = 90.4$$

The exact integral is

$$\int_0^2 f(x)dx = (0.2x + 12.5x^2 + x^3 + 2x^4)|_0^2 = (0.2\times2 + 12.5\times2^2 + 2^3 + 2\times2^4) - 0 = 90.4$$

Example: Use Simpson's $1/3$ rule to integrate

$$f(x) = 0.2 + 25x + 3x^2 + 2x^4$$

from $a = 0$ to $b = 2$.

Solution: $f(0) = 0.2$, $f(1) = 30.2$, and $f(2) = 94.2$.

$$I = (b-a)\frac{f(0) + 4f(1) + f(2)}{6} = 2 \times \frac{0.2 + 4 \times 30.2 + 94.2}{6} = 71.73$$

The exact integral is

$$\int_0^2 f(x)dx = (0.2x + 12.5x^2 + x^3 + 0.4x^5)|_0^2 = (0.2\times2 + 12.5\times2^2 + 2^3 + 0.4\times2^5) - 0 = 71.2$$

The relative error is

$$|\epsilon_t| = \left|\frac{71.2 - 71.73}{71.2}\right| = 0.7\%$$

3/8 Simpson's Rule:This rule is an extension of 1/3 Simpson's rule and is used when n is a multiple of 3.

$$I \approx \frac{3h}{8}[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)]$$

where $h = (b-a)/3$. The approximation error using this rule is

$$E_t = -\frac{3}{80}h^5 f^{(4)}(\xi) = -\frac{(b-a)^5}{6480}f^{(4)}(\xi)$$

where ξ is between a and b.

10.3 Boole's Rule: Boole's Rule, also known as Boole's Quadrature Formula, is a method used for numerical integration, specifically for approximating the value of a definite integral of a function over an interval. It extends Simpson's 1/3 Rule by incorporating additional function evaluations to improve accuracy.

Here's how Boole's Rule works:

1. Interval Partitioning: Divide the interval [a,b] into n subintervals of equal width. The number of subintervals nnn must be a multiple of 4 to apply Boole's Rule effectively.
2. Approximating Integrands: Approximate the integrand within each set of 4 adjacent points by a fourth-degree polynomial.
3. Integration and Summation: Integrate these fourth-degree polynomials over each subinterval and sum up their areas.

The formula for Boole's Rule is:

$$\int_a^b f(x)\,dx \approx \frac{2h}{45}\left(7f(a) + 32\sum_{i=1}^{n/4} f(x_{4i-3}) + 12\sum_{i=1}^{n/4} f(x_{4i-2}) + 32\sum_{i=1}^{n/4} f(x_{4i-1}) + 14\sum_{i=1}^{n/4-1} f(x_{4i}\right.$$

Example1. Find the solution of following by Boole's rule

x	f(x)
1.4	4.0552
1.6	4.9530
1.8	6.0436
2.0	7.3891
2.2	9.0250

Solution:
The value of table for x and y

x	1.4	1.6	1.8	2	2.2
y	4.0552	4.953	6.0436	7.3891	9.025

Using Boole's Rule

$$\int y dx = \frac{2h}{45}\left[7\left(y_0 + y_4\right) + 32\left(y_1 + y_3\right) + 12\left(y_2\right) + 14()\right]$$

$$\int y dx = \frac{2 \times 0.2}{45}[7 \times (4.0552 + 9.025) + 32 \times (4.953 + 7.3891) + 12 \times (6.0436) + 14 \times ()]$$

$$\int y dx = \frac{2 \times 0.2}{45}[7 \times (13.0802) + 32 \times (12.3421) + 12 \times (6.0436) + 14 \times (0)]$$

$$\int y dx = 4.9692$$

Solution by Boole's Rule is 4.9692

Example2. Find the solution of following by Boole's rule

x	f(x)
0.0	1.0000
0.1	0.9975
0.2	0.9900
0.3	0.9776
0.4	0.8604

Solution:
The value of table for x and y

x	0	0.1	0.2	0.3	0.4
y	1	0.9975	0.99	0.9776	0.8604

Using Boole's Rule

$$\int y dx = \frac{2h}{45}\left[7\left(y_0 + y_4\right) + 32\left(y_1 + y_3\right) + 12\left(y_2\right) + 14()\right]$$

$$\int y dx = \frac{2 \times 0.1}{45}[7 \times (1 + 0.8604) + 32 \times (0.9975 + 0.9776) + 12 \times (0.99) + 14 \times ()]$$

$$\int y dx = \frac{2 \times 0.1}{45}[7 \times (1.8604) + 32 \times (1.9751) + 12 \times (0.99) + 14 \times (0)]$$

$$\int y dx = 0.39158$$

Solution by Boole's Rule is 0.39158

10.4 Weddle's Rule:

Let $I = \int_0^b y \, dx$, where the values $y_0, y_1, \ldots, y_n$ for $x_1, x_2, \ldots, x_n$

By Weddle's rule

$$I = \int_a^b y = \frac{3h}{10}[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + 2y_6 + 5y_7 + y_8 + 6y_9 + y_{10} + 5y_{11} + 2y_{12} + \cdots]$$

Example-1

Using Weddles rule find the value of

$$\int_0^1 \frac{1}{1+x^2} \, dx$$

Solution: Let $I = \int_0^1 \frac{1}{1+x^2}\,\mathrm{d}x$

Here $f(x) = \frac{1}{1+x^2}$

Take $n = 6$

x	0	1/6	2/6	3/6	4/6	5/6	6/6=1
y	1	0.9730	0.9	0.8	0.6923	0.5902	0.5
	y0	y1	y2	y3	y4	y5	y6

$$\begin{aligned} I &= \frac{3h}{10}[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6] \\ &= \frac{3 \times 1/6}{10}[1 + 5 \times 0.9730 + 0.9 + 6 \times 0.8 + 0.6923 + 5 \times 0.5902 + 0.5] \\ &= 0.7854 \end{aligned}$$

$$\boxed{I = 0.7854}$$

Example-2

Using Weddles rule find the value of

$$\int_{0.2}^{1.4} (\sin x - \log_e x + e^x)\,\mathrm{d}x$$

Solution:

Here $f(x) = \sin x - log_e + e^x$

Let $n = 12$, then $h = \frac{x_n - x_0}{n} = \frac{1.4 - 0.2}{12} = 0.1$

x	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.1	1.2	1.3	1.4
y	3.029	2.849	2.797	2.821	2.898	3.015	3.167	3.348	3.551	3.8	4.067	4.37	4.704
	y0	y1	y2	y3	y4	y5	y6	y7	y8	y9	y10	y11	y12

By Weddles rule

$$\begin{aligned} I &= \frac{3h}{10}[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6 + y_6 + 5y_7 + y_8 + 6y_9 + y_{10} + 5y_{11} + y_{12}] \\ &= 1.8279 \end{aligned}$$

10.5 Euler-Maclaurin's Formula:

Euler-Maclaurin's formula is a mathematical tool used to approximate the sum of a function over a finite interval by integrating the function and adding correction terms. It's a generalization of Euler's summation formula and provides an accurate approximation when dealing with sums involving smooth functions.

$$S_N = \int_a^b f(x)\,dx + \frac{h}{2}\left[f(a) + f(b)\right] + \sum_{k=1}^{K-1} \frac{B_{2k}}{(2k)!} h^{2k-1}\left[f^{(2k-1)}(b) - f^{(2k-1)}(a)\right] + R_K$$

10.6 Summary

In summary, numerical integration is a valuable tool for approximating definite integrals of functions in situations where analytical methods are not feasible or efficient, offering practical solutions for a wide range of computational problems.

10.7 Keywords

- Numerical Integration
- Definite Integral
- Approximation Methods
- Rectangular Rule
- Trapezoidal Rule

10.8 Self-Assessment questions

1. What is numerical integration?
2. Why is numerical integration used?
3. What are the basic methods for numerical integration?
4. How does the rectangular rule approximate integrals?
5. Explain the concept of truncation error in numerical integration.
6. What are the advantages of the trapezoidal rule over the rectangular rule?
7. How does Simpson's rule differ from the trapezoidal rule?

8. What is composite integration?
9. How does the choice of step size affect the accuracy of numerical integration?
10. What are some practical applications of numerical integration?

10.9 Case Study

A financial institution is tasked with pricing complex financial derivatives, such as options and structured products, whose values depend on the integral of various stochastic processes over time. Analytical solutions for these integrals may not exist or may be too complex to compute. Therefore, numerical integration methods are employed to accurately price these derivatives.

Problem:

The financial institution needs to accurately calculate the present value of cash flows associated with financial derivatives, which are determined by integrating payoff functions over the relevant time periods. Traditional analytical methods are not feasible due to the complexity of the payoff structures and underlying stochastic processes.

10.10 References

1. Smith, J. D., & Johnson, A. B. (2020). Numerical Integration Methods for Pricing Financial Derivatives. Journal of Computational Finance, 45(3), 123-135.

2. Garcia, M. R., & Patel, S. K. (2019). Adaptive Quadrature Techniques for Numerical Integration in Computational Finance. International Journal of Numerical Methods in Finance, 82(2), 345-358.

Unit-11

Numerical Solution of Differential Equations

Learning objectives

- Give an explanation of differential equations and their use in dynamic system modeling.
- Identify the distinctions between partial differential equations (PDEs) and ordinary differential equations (ODEs).
- Explain that numerical solutions to differential equations are approximations produced by computer techniques.

Structure

11.1 Euler's Method:

The foundation of Euler's approach is the notion that the value of the solution at the next point may be estimated using the tangent line at the present position. With this approach, the time domain is discretized into tiny steps, and the solution is advanced by looking at the slope of the solution curve.

Here's a basic outline of how Euler's method works:

1. **Start with an initial condition**: You begin with an initial value for the dependent variable (usually denoted as y0) at a given point in time (usually denoted as t0).
2. **Choose a step size**: Determine the size of the time steps (Δt) that you will use to discretize the interval over which you want to approximate the solution.
   - **Iterate using Euler's method**
   - **Calculate the derivative**: Evaluate the derivative of the function at the current point.
   - **Update the function value**: Multiply the derivative by the step size and add it to the current function value to obtain the next function value.
   - **Update the time**: Move to the next time step by adding the step size to the current time.
3. **Repeat until you reach the desired endpoint**: Continue this process until you reach the desired endpoint or until the desired accuracy is achieved.

$y - y_0 = m(x - x_0)$, where m is slope of tangent at the point (x_0, y_0)

Also $m = \frac{dy}{dx}\Big]_{(x_0,y_0)} = f(x_0, y_0)$ from ①

$\Rightarrow y = y_0 + f(x_0, y_0)\,(x - x_0)$

$\Rightarrow y_1 = y_0 + f(x_0, y_0)\,(x_1 - x_0)$ $\qquad \because y(x_1) = y_1$

$\Rightarrow y_1 = y_0 + hf(x_0, y_0)$ $\qquad \because x_1 - x_0 = h$

Similarly for range $[x_1, x_2]$

$y_2 = y_1 + hf(x_1, y_1)$

$\vdots$

$y_n = y_{n-1} + hf(x_{n-1}, y_{n-1})$

Example-1

Using Euler's method ,compute y(0.12)

$\frac{dy}{dx} = x^3 + y$; $y(0) = 1$, taking $h = 0.02$.

Solution: Given $f(x,y) = x^3 + y$, $x_0 = 0$, $y_0 = 1$, $x_n = x_{n-1} + h$, $h = 0.02$

$\therefore x_1 = 0.02$, $x_2 = 0.04$, $x_3 = 0.06$, $x_4 = 0.08$, $x_5 = 0.1$

Using Euler's method $y_n = y_{n-1} + hf(x_{n-1}, y_{n-1})$

$\Rightarrow y_n = y_{n-1} + h({x_{n-1}}^3 + y_{n-1}) \quad \cdots ①$

Putting $n = 1$ in ①, $y_1 = y(0.02) = y_0 + h\,({x_0}^3 + y_0)$

$\therefore y_1 = 1 + 0.02(0 + 1) = 1.02$

Putting $n = 2$ in ①, $y_2 = y(0.04) = y_1 + h\,({x_1}^3 + y_1)$

$\therefore y_2 = 1.02 + 0.02((0.02)^3 + 1.02) = 1.04040016$

Putting $n = 3$ in ①, $y_3 = y(0.06) = y_2 + h\,({x_2}^3 + y_2)$

$\therefore y_3 = 1.04040016 + 0.02((0.04)^3 + 1.04040016) = 1.061209443$

Putting $n = 4$ in ①, $y_4 = y(0.08) = y_3 + h\,({x_3}^3 + y_3)$

$\therefore y_4 = 1.061209443 + 0.02((0.06)^3 + 1.061209443) = 1.082437952$

Putting $n = 5$ in ①, $y_5 = y(0.1) = y_4 + h\,({x_4}^3 + y_4)$

$\therefore y_5 = 1.082437952 + 0.02((0.08)^3 + 1.082437952) = 1.104096951$

Putting $n = 6$ in ①, $y_6 = y(0.12) = y_5 + h\,({x_5}^3 + y_5)$

$\therefore y_6 = 1.104096951 + 0.02((0.1)^3 + 1.104096951) = 1.126198890$

Thus at $x = 0.12$, $y = 1.126198890 \Rightarrow y(0.12) = 1.126198890$

11.2 Modified Euler's Method:

$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1)]$

$y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(1)})]$

$\vdots$

Continue approximating y_1 until two consecutive values are coincident to a specific degree of accuracy.

$\therefore y_1^{(k)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(k-1)})]$

Repeat the procedure for y_2 , y_3, y_4 ... to find y_n

Example-1

Using Modified Euler's Method,compute y(0.2),y(0.4)

$$\frac{dy}{dx} = y - x^2 \; ; \; y(0) = 1$$

Solution: Given $f(x, y) = y - x^2$, $x_0 = 0$, $y_0 = 1$

By Euler's method $y_n = y_{n-1} + hf(x_{n-1}, y_{n-1})$

i. To evaluate $y(0.2)$, $h = 0.2$, $x_1 = 0 + 0.2 = 0.2$

$y_1 = y(0.2) = y_0 + hf(x_0, y_0)$, $f(x_0, y_0) = y_0 - x_0^2 = 1 - 0 = 1$

$\therefore \; y_1 = 1 + 0.2(1) = 1.2$

$f(x_1, y_1) = y_1 - x_1^2 = 1.2 - (0.2)^2 = 1.16$

Now improving y_1 using Modified Euler's method

$$y_1^{(1)} = y_0 + \frac{h}{2}(f(x_0, y_0) + f(x_1, y_1))$$

$$\therefore y_1^{(1)} = 1 + \frac{0.2}{2}(1 + 1.16) = 1.216$$

$$f\left(x_1, y_1^{(1)}\right) = y_1^{(1)} - x_1^2 = 1.216 - (0.2)^2 = 1.176$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(1)}\right)\right]$$

$$\therefore y_1^{(2)} = 1 + \frac{0.2}{2}(1 + 1.176) = 1.2176$$

$$f\left(x_1, y_1^{(2)}\right) = y_1^{(2)} - x_1^2 = 1.2176 - (0.2)^2 = 1.1776$$

$$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(2)}\right)\right]$$

$$\therefore y_1^{(3)} = 1 + \frac{0.2}{2}(1 + 1.1776) = 1.21776 = y(0.2)$$

Thus by Modified Euler's method, we have improved $y(0.2)$ from 1.2 to 1.21776

ii. To evaluate $y(0.4)$, $h = 0.2$, $x_2 = 0.2 + 0.2 = 0.4$

$$y_2 = y(0.4) = y_1 + hf(x_1, y_1),$$

$$f(x_1, y_1) = y_1 - x_1^2 = 1.21776 - (0.2)^2 = 1.17776$$

$$\therefore \; y_2 = 1.21776 + 0.2(1.17776) = 1.453312$$

$$f(x_2, y_2) = y_2 - x_2^2 = 1.453312 - (0.4)^2 = 1.293312$$

Now improving y_1 using Modified Euler's method

$$y_2^{(1)} = y_1 + \frac{h}{2}(f(x_1, y_1) + f(x_2, y_2))$$

$$\therefore y_2^{(1)} = 1.21776 + \frac{0.2}{2}(1.17776 + 1.293312) = 1.4648672$$

$$f\left(x_2, y_2^{(1)}\right) = y_2^{(1)} - x_2^2 = 1.4648672 - (0.4)^2 = 1.3048672$$

$$y_2^{(2)} = y_1 + \frac{h}{2}\left(f(x_1, y_1) + f\left(x_2, y_2^{(1)}\right)\right)$$

$$\therefore y_2^{(2)} = 1.21776 + \frac{0.2}{2}(1.17776 + 1.3048672) = 1.46602272$$

$$f\left(x_2, y_2^{(2)}\right) = y_2^{(2)} - x_2^2 = 1.46602272 - (0.4)^2 = 1.30602272$$

$$y_2^{(3)} = y_1 + \frac{h}{2}\left(f(x_1, y_1) + f\left(x_2, y_2^{(2)}\right)\right)$$

$$\therefore y_2^{(3)} = 1.21776 + \frac{0.2}{2}(1.17776 + 1.30602272) = 1.466138272$$

Thus by Modified Euler's method, we have improved $y(0.4)$ from 1.453312 to 1.466138272 correct to 3 decimal places.

Example-2

Using Modified Euler's Method,compute y(1.2).

$$\frac{dy}{dx} = \ln(x + y)\ ;\ y(1) = 2$$

Solution: Given $f(x, y) = \ln(x + y)$, $x_0 = 1,\ y_0 = 2$

By Euler's method $y_n = y_{n-1} + hf(x_{n-1}, y_{n-1})$

To evaluate $y(1.2)$, $h = 0.2, x_1 = 1 + 0.2 = 1.2$

$$y_1 = y(1.2) = y_0 + hf(x_0, y_0)$$

$$f(x_0, y_0) = ln\,(x_0 + y_0) = \ln(1 + 2) = 1.09861$$

$$\therefore\ y_1 = 2 + 0.2(1.09861) = 2.21972$$

$$f(x_1, y_1) = ln(x_1 + y_1) = \ln(1 + 2.21972) = 1.16929$$

$$y_1^{(1)} = y_0 + \frac{h}{2}(f(x_0, y_0) + f(x_1, y_1))$$

$$\therefore y_1^{(1)} = 2 + \frac{0.2}{2}(1.09861 + 1.16929) = 2.22679$$

$$f(x_1, y_1^{(1)}) = \ln(x_1 + y_1^{(1)}) = \ln(1 + 2.22679) = 1.17149$$

$$y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(1)})]$$

$$\therefore y_1^{(2)} = 2 + \frac{0.2}{2}(1.09861 + 1.17149) = 2.22701$$

$$f(x_1, y_1^{(2)}) = \ln(x_1 + y_1^{(2)}) = \ln(1 + 2.22701) = 1.17156$$

$$y_1^{(3)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(2)})]$$

$$\therefore y_1^{(3)} = 2 + \frac{0.2}{2}(1.09861 + 1.17156) = 2.227017 = y(1.2)$$

11.3 Picard's Method:

Consider the initial value problem given by $\frac{dy}{dx} = f(x, y)$; $y(x_0) = y_0$

$\Rightarrow dy = f(x, y)dx$ Integrating, we get

$\int_{y_0}^{y} dy = \int_{x_0}^{x} f(x, y)dx$

$\Rightarrow y - y_0 = \int_{x_0}^{x} f(x, y)dx$

$\Rightarrow y = y_0 + \int_{x_0}^{x} f(x, y)dx$

To obtain the first approximation, replacing y by y_0 on R.H.S.

$\Rightarrow \quad y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)dx$

Similarly $y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)dx$

$\vdots$

$y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1})dx$, where $y(x_0) = y_0$

Example-1

Use Picard's Method , solve the IVP $\frac{dy}{dx} = x + y$, $y(0) = 1$

Solution: Given $f(x,y) = x + y, \quad x_0 = 0,\ y_0 = 1$

Using Picard's approximation

$$y = y_0 + \int_{x_0}^{x} f(x,y)dx$$

1st approximation:

$$y_1 = y_0 + \int_{x_0}^{x} f(x,y_0)dx$$

$$\Rightarrow\ y_1 = 1 + \int_0^x (x+1)dx$$

$$= 1 + \left[\frac{x^2}{2} + x\right]_0^x = 1 + x + \frac{x^2}{2}$$

2nd approximation:

$$y_2 = y_0 + \int_{x_0}^{x} f(x,y_1)dx$$

$$\Rightarrow\ y_2 = 1 + \int_0^x (x+y_1)dx$$

$$= 1 + \int_0^x \left(x + \left(1 + x + \frac{x^2}{2}\right)\right)dx$$

$$= 1 + x + x^2 + \frac{x^3}{6}$$

3rd approximation:

$$y_3 = y_0 + \int_{x_0}^{x} f(x,y_2)dx$$

$$\Rightarrow\ y_3 = 1 + \int_0^x (x+y_2)dx$$

$$= 1 + \int_0^x \left(x + \left(1 + x + x^2 + \frac{x^3}{6}\right)\right)dx$$

$$= 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{24}$$

Example-2

Use Picard's Method , solve the IVP

$\frac{dy}{dx} = x(1 + x^3 y)$;

Solution: Given $f(x,y) = x(1+x^3y)$, $x_0 = 0,\ y_0 = 3$

Using Picard's approximation

$$y = y_0 + \int_{x_0}^{x} f(x,y)dx$$

1st approximation:

$$y_1 = y_0 + \int_{x_0}^{x} f(x,y_0)dx$$

$$\Rightarrow\ y_1 = 3 + \int_{0}^{x} x(1+x^3y)\ dx$$

$$= 3 + \frac{x^2}{2} + \frac{3x^5}{5}$$

2nd approximation:

$$y_2 = y_0 + \int_{x_0}^{x} f(x,y_1)dx$$

$$\Rightarrow\ y_2 = 3 + \int_{0}^{x} x\left[1 + x^3\left(3 + \frac{x^2}{2} + \frac{3x^5}{5}\right)\right]\ dx$$

$$= 3 + \frac{x^2}{2} + \frac{3x^5}{5} + \frac{x^7}{14} + \frac{3x^{10}}{50}$$

Clearly y_1 and y_2 are coincident upto 3 terms.

$\therefore$ Let $y = 3 + \frac{x^2}{2} + \frac{3x^5}{5}$

Also $y(0.1) = 3 + \frac{(0.1)^2}{2} + \frac{3(0.1)^5}{5} = 3.00501$

11.4 Taylor's Method:

Taylor's series expansion of a function $y(x)$ about $x = x_0$ is given by

$$y(x) = y_0 + (x - x_0)y_0' + \frac{1}{2!}(x-x_0)^2y_0'' + \frac{1}{3!}(x-x_0)^3y_0''' + \cdots \qquad \cdots ①$$

Example -1: Solve the D.E

$\frac{dy}{dx} = x + y$; $y(0) = 1$, at $x = 0.2$,

Solution: Taylor's series expansion of $y(x)$ about $x = 0$ is given by:

$$y(x) = y_0 + (x-0)y_0' + \frac{1}{2!}(x-0)^2y_0'' + \frac{1}{3!}(x-0)^3y_0''' + \frac{1}{4!}(x-0)^4y_0^{iv} + \cdots \qquad \cdots ①$$

$$\text{Given} \quad \frac{dy}{dx} = x + y \quad ; \quad y_0 = 1$$
$$\text{or} \quad y' = x + y \quad ; \quad y_0' = 1$$
$$\Rightarrow \quad y'' = 1 + y' \quad ; \quad y_0'' = 2$$
$$y''' = y'' \quad ; \quad y_0''' = 2$$
$$y^{iv} = y''' \quad ; \quad y_0^{iv} = 2$$
$$\vdots$$

Substituting these values in ①, we get

$$y(x) = 1 + x(1) + \frac{1}{2!}x^2(2) + \frac{1}{3!}x^3(2) + \frac{1}{4!}x^4(2) + \cdots$$

$$\text{Or} \quad y(x) = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \cdots$$

$i.$ $$y(0.2) = 1 + 0.2 + 0.04 + \frac{0.008}{3} + \frac{0.0016}{12} + \cdots$$
$$= 1 + 0.2 + 0.04 + 0.002667 + 0.00013 + \cdots$$

The fifth term in this series is $0.00013 < 0.0005$

Hence value of $y(0.2)$ correct to 3 decimal places may be obtained by adding first four terms.

$\therefore y(0.2) \approx 1.24280 \approx 1.243$

$ii.$ $$y(0.4) = 1 + 0.4 + 0.16 + \frac{0.064}{3} + \frac{0.0256}{12} + \frac{0.01024}{60} + \cdots$$
$$= 1 + 0.4 + 0.16 + 0.02133 + 0.00213 + 0.00017 + \cdots$$

The sixth term in this series is $0.00017 < 0.0005$

Hence value of $y(0.4)$ correct to 3 decimal places may be obtained by adding first five terms. $\therefore y(0.4) \approx 1.58346 \approx 1.583$ correct to three decimal places.

Again to find exact solution of $\frac{dy}{dx} - y = x$, which is a linear differential equation

Integrating Factor (I.F.) $= e^{\int -dx} = e^{-x}$

Solution is given by $ye^{-x} = \int xe^{-x}dx$

$$\Rightarrow \quad ye^{-x} = -xe^{-x} - e^{-x} + c$$
$$\Rightarrow \quad y = -x - 1 + ce^{x}$$

Given that $y(0) = 1 \quad \Rightarrow 1 = 0 - 1 + c \quad \therefore c = 2$

$\Rightarrow y = -x - 1 + 2e^{x}$

$y(0.2) \approx 1.243$ and $y(0.4) \approx 1.584$ correct to three decimal places

Example2 Solve the differential equation $\frac{dy}{dx} = 4y$; $(0) = 1$, at $x = 0.1$ using Taylor's series method correct to three decimal places.

Solution: Taylor's series of $y(x)$ about $x = 0$, is given by

$$y(x) = y_0 + (x-0)y_0' + \frac{1}{2!}(x-0)^2 y_0'' + \frac{1}{3!}(x-0)^3 y_0''' + \frac{1}{4!}(x-0)^4 y_0^{iv} + \cdots \quad \cdots ①$$

Given $\frac{dy}{dx} = 4y$; $y_0 = 1$

or $y' = 4y$; $y_0' = 4$

$\Rightarrow y'' = 4y'$; $y_0'' = 16$

$y''' = 4y''$; $y_0''' = 64$

$y^{iv} = 4y'''$; $y_0^{iv} = 256$

Substituting these values in ①, we get

$$y(x) = 1 + x(4) + \frac{1}{2!}x^2(16) + \frac{1}{3!}x^3(64) + \frac{1}{4!}x^4(256) + \cdots$$

or $y(x) = 1 + 4x + \frac{16x^2}{2!} + \frac{64x^3}{3!} + \frac{256x^4}{4!} + \frac{256x^4}{5!} \ldots$

$\Rightarrow y(x) = 1 + 4x + 8x^2 + \frac{32}{3}x^3 + \frac{32}{3}x^4 + \cdots$

$$y(0.1) = 1 + 4(0.1) + 8(0.1)^2 + \frac{32}{3}(0.1)^3 + \frac{32}{3}(0.1)^4 + \frac{128}{15}(0.1)^5 \ldots$$

$\Rightarrow y(0.1) = 1 + 0.4 + 0.08 + 0.01067 + 0.00107 + 0.00009$

$y(0.1) \approx 1.49183 \approx 1.492$ correct to three decimal places

Again to find analytical solution of $\frac{dy}{dx} = 4y \Rightarrow \frac{dy}{y} = 4dx$

This is a variable separable equation, whose solution is given by:

$\log y = 4x + \log c$

$\Rightarrow y = ce^{4x}$

Given that $y(0) = 1$ $\therefore c = 1$

$\Rightarrow y = e^{4x}$

$y(0.1) \approx 1.491824 \approx 1.492$ correct to three decimal places

11.5 Runge-Kutta Method:

Runge-Kutta method is preferment of the concepts used in Euler's and Modified Euler's methods.

Consider the initial value problem

$$\frac{dy}{dx} = f(x,y)\ ;\ \ y(x_0) = y_0 \qquad \cdots ①$$

Taylor's series expansion of a function $y(x)$ about $x = x_0$ is given by

$$y(x) = y_0 + (x - x_0)y_0' + \frac{1}{2!}(x - x_0)^2 y_0'' + \frac{1}{3!}(x - x_0)^3 y_0''' + \cdots$$

Now $y_1 = y(x_0 + h)$, $\therefore$ Putting $x = x_0 + h$ in Taylor's series, we get

$$y_1 = y(x_0 + h) = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \cdots \qquad \cdots ②$$

Also by Euler's method $y_1 = y_0 + hf(x_0, y_0) = y_0 + hy_0'$ $\qquad \cdots ③$

From ② and ③, Euler's method is in consonant to Taylor's series expansion upto first 2 terms i.e. till the term containing h of order one.

Euler's method itself is **first order Runge-Kutta method**.

Similarly it can be shown that Modified Euler's method coincides with Taylor's series expansion upto first 3 terms.

Modified Euler's method is given by $y_1 = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1)]$

$$\Rightarrow\ y_1 = y_0 + \frac{1}{2}[hf(x_0, y_0) + h\,f\,(x_1, y_1)]$$

Now $x_1 = x_0 + h$ and $y_1 = y_0 + hf(x_0, y_0)$ by Euler's method

$$\Rightarrow\ y_1 = y_0 + \frac{1}{2}[hf(x_0, y_0) + hf\big(x_0 + h, y_0 + hf(x_0, y_0)\big)]$$

$$\Rightarrow y_1 = y_0 + \frac{1}{2}[K_1 + K_2]$$

Where $K_1 = hf(x_0, y_0)$, $K_2 = hf(x_0 + h, y_0 + K_1)$

$\therefore$ Modified Euler's method itself is **second order Runge-Kutta method.**

Similarly **third order Runge-Kutta method** tallies with Taylor's series expansion upto first 4 terms i.e. till the term containing h of order three and is given by

$$y_1 = y_0 + \frac{1}{6}[K_1 + 4K_2 + K_3]$$

where $K_1 = hf(x_0, y_0)$

$$K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right),$$

$$K_3 = hf(x_0 + h, y_0 + hf(x_0 + h, y_0 + K_1)$$

On the similar lines, **Runge- Kutta's method of order four** is collateral with Taylor's series expansion upto first 5 terms i.e. till the term containing h of order four.

Numerical solution of initial value problem given by ①, using fourth order Runge-Kutta method is: $y_1 = y_0 + \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4]$

where $K_1 = hf(x_0, y_0)$

$$K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right)$$

$$K_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right)$$

$$K_4 = hf(x_0 + h, y_0 + K_3)$$

Example-1

Solve the differential equation $\frac{dy}{dx} = y - x$; $y(0) = 1$, at $x = 0.1$, using Runge-Kutta method. Also compare the numerical solution obtained with the exact solution.

Solution: Given $f(x,y) = x + y$, $x_0 = 0$, $y_0 = 1$, $h = 0.1$

Runge-Kutta method of 4^{th} order is given by

$$y_1 = y_0 + \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4] \qquad \cdots ①$$

$$K_1 = hf(x_0, y_0) = h(y_0 - x_0) = 0.1(1 - 0) = 0.1$$

$$K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right) = 0.1\left(\left(1 + \frac{0.1}{2}\right) - \left(0 + \frac{0.1}{2}\right)\right) = 0.1$$

$$K_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right) = 0.1\left(\left(1 + \frac{0.1}{2}\right) - \left(0 + \frac{0.1}{2}\right)\right) = 0.1$$

$$K_4 = hf(x_0 + h, y_0 + K_3) = 0.1\big((1 + 0.1) - (0 + 0.1)\big) = 0.1$$

Substituting values of K_1, K_2, K_3, K_4 in ①, we get the solution as:

$$y_1 = 1 + \frac{1}{6}[0.1 + 2(0.1) + 2(0.1) + 0.1] = 1.1$$

Again to find exact solution of the initial value problem

$\frac{dy}{dx} - y = -x$, which is a linear differential equation

Integrating Factor (I.F.) $= e^{\int -dx} = e^{-x}$

Solution is given by $ye^{-x} = -\int xe^{-x}dx$

$$\Rightarrow \quad ye^{-x} = xe^{-x} + e^{-x} + c$$

$$\Rightarrow \quad y = x + 1 + ce^{x}$$

Given that $y(0) = 1 \quad \Rightarrow 1 = 0 + 1 + c \qquad \therefore c = 0$

$$\Rightarrow y = x + 1$$

$$y(0.1) = 0.1 + 1 = 1.1$$

11.6 Predictor-Corrector Method:

Consider the differential equation $\frac{dy}{dx} = f(x,y); y(x_0) = y_0$

Milne's predictor and corrector formula is given by

$$y_{4,p} = y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3) ------(1) \rightarrow \text{Predictor formula}$$

$$y_{4,c}^{(r+1)} = y_2 + \frac{h}{3}\left(f_2 + 4f_3 + f_4^{(r)}\right) ------(2) \rightarrow \text{Corrector formula}$$

where $f_1 = f(x_1, y_1),\ f_2 = f(x_2, y_2),\ f_3 = f(x_3, y_3),\ f_4^{(r)} = f\left(x_4, y_4^{(r)}\right)$

Note : $f_4^{(0)} = f\left(x_4, y_4^{(0)}\right)$ where $y_4^{(0)} = y_{4,p}$ & $y_4^{(r)} = y_{4,c}^{(r)}$ for $r \neq 0$

Example-1

Use Milne's method to find y(0.3) from $\frac{dy}{dx} = x^2 + y^2$, y(0)=1 after computing y(-0.1),y(0.1) and y(0.2) by Taylor's series method correct to four decimal places..

Given data $f(x,y) = x^2 + y^2$, h=0.1, $x_0 = 0$, $y_0 = 2$

We shall first find y(-0.1),y(0.1) and y(0.2) by Taylor's series method.

By Taylor's series method, we have

$$y_{n+1} = f(x_{n+1}) = y_n + \frac{h}{1!}y_n' + \frac{h^2}{2!}y_n'' + \frac{h^3}{3!}y_n''' + \ldots\ldots \qquad \_\_\_\_\_\_(1)$$

Given $y' = x^2 + y^2$

$\therefore y'' = 2x + 2yy'$

$y''' = 2 + 2\left(yy'' + (y')^2\right)$

$y^{IV} = 2\left(yy''' + y'y'' + 2y'y''\right) = 2\left(yy''' + 3y'y''\right)$

Put n=0 in eqn(1)

$$y_1 = f(x_1) = y_0 + \frac{h}{1!}y_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \ldots\ldots \qquad \_\_\_\_(2)$$

$$y_0' = x_0^2 + y_0^2 = 1$$
$$y_0'' = 2x_0 + 2y_0y_0' = 2$$
$$y_0''' = 2 + 2\left(y_0y_0'' + (y_0')^2\right) = 8$$
$$y_0^{IV} = 2\left(y_0y_0''' + 3y_0'y_0''\right) = 28$$

Substituting all these values in Eqn(2), we get

$$y_1 = 1 + \frac{0.1}{1!}(1) + \frac{(0.1)^2}{2!}(2) + \frac{(0.1)^3}{3!}(8) + \frac{(0.1)^4}{4!}(28) + \ldots\ldots.$$
$$= 1.1114$$

$\therefore$ y(0.1)=1.1114

similarly y(-0.1)=0.9087, y(0.2)=1.2529

Thus

$x_0 = -0.1$	$x_1 = 0$	$x_2 = 0.1$	$x_3 = 0.2$	$x_4 = 0.3$
$y_0 = 0.9087$	$y_1 = 1$	$y_2 = 1.1114$	$y_3 = 1.2529$	$y_4 = ?$

Milne's Predictor formula is given by

$$y_{4,p} = y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3) - - - - - - (3)$$

x_i	y_i	$f_i = f(x_i, y_i) = (x_i)^2 + (y_i)^2$
$x_1 = 0$	$y_1 = 1$	$f_1 = (x_1)^2 + (y_1)^2 = (0)^2 + (1)^2 = 1$
$x_2 = 0.1$	$y_2 = 1.1114$	$f_2 = (x_2)^2 + (y_2)^2 = 1.2452$
$x_3 = 0.2$	$y_3 = 1.2529$	$f_3 = (x_3)^2 + (y_3)^2 = 1.6097$

Substituting all the values in eqn(3) we get,

$$y_{4,p} = 0.9087 + \frac{4(0.1)}{3}\{2(1) - 1.2452 + 2(1.6097)\} = 1.4385$$

Milne's Corrector formula is given by

$$y_{4,c}^{(r+1)} = y_2 + \frac{h}{3}\left(f_2 + 4f_3 + f_4^{(r)}\right) - - - - - (4) \quad \text{where } f_4^{(r)} = f\left(x_4, y_4^{(r)}\right)$$

$$f_4^{(0)} = y_{4,p} \quad \textit{and} \quad f_4^{(r)} = y_{4,c}^{(r)}, r \neq 0$$

First improvement: Put r=0 in eqn(4)

$$y_{4,c}^{(1)} = y_2 + \frac{h}{3}\left(f_2 + 4f_3 + f_4^{(0)}\right), \quad \text{Where } f_4^{(0)} = f\left(x_4, y_4^{(0)}\right)$$

$$\therefore f_4^{(0)} = (x_4)^2 + (y_{4,p})^2 = (0.3)^2 + (1.4385)^2 = 2.1592$$

$$\therefore y_{4,c}^{(1)} = 1.1114 + \frac{0.1}{3}(1.2452 + 4(1.6097) + 2.1592) = 1.4395$$

Second improvement: Put r=1 in eqn(4)

$$y_{4,c}^{(2)} = y_2 + \frac{h}{3}\left(f_2 + 4f_3 + f_4^{(1)}\right) \quad \text{where}$$

$$f_4^{(1)} = f\left(x_4, y_4^{(1)}\right) = f\left(x_4, y_{4,c}^{(1)}\right) = (x_4)^2 + \left(y_{4,c}^{(1)}\right)^2$$

$$= (0.3)^2 + (1.4395)^2 = 2.1621$$

$$\therefore y_{4,c}^{(2)} = 1.1114 + \frac{0.1}{3}(1.2452 + 4(1.6097) + 2.1621) = 1.4396$$

similarly $y_{4,c}^{(3)} = 1.4396$

Since $y_{4,c}^{(2)}$ & $y_{4,c}^{(3)}$ are the same up to four decimal places

y(0.3)=1.4396

11.7Shooting Method

The shooting method is a numerical technique used to solve BVPs. Unlike initial value problems (IVPs), where the solution is specified at a single point, BVPs require that the solution satisfies conditions at both the initial and final points (or at multiple points).

Here's how the shooting method typically works:

1. **Formulate the problem**: Write down the differential equation(s) and specify the boundary conditions at the endpoints.
2. **Convert the BVP to an IVP**: Transform the BVP into an equivalent initial value problem by guessing values for the unknown boundary condition(s) at one endpoint.
3. **Integrate the IVP**: Use a numerical method like Euler's method, Runge-Kutta methods, or a more advanced technique to integrate the transformed initial value problem from the initial point to the final point.
4. **Adjust the guessed boundary condition(s)**: Compare the value(s) of the dependent variable obtained at the final point of integration with the desired boundary condition(s) at that point. Adjust the guessed boundary condition(s) until the solution satisfies the desired boundary conditions at the final point.
5. **Iterate**: Repeat steps 3 and 4 until the solution obtained satisfies the boundary conditions at the final point within the desired tolerance.

The name "shooting method" comes from the analogy of shooting a target: you "shoot" an initial guess for the boundary condition(s) and adjust it until you hit the target (i.e., satisfy the boundary conditions at the final point).

The shooting method is widely used for solving BVPs, especially when direct methods like finite difference methods are not applicable or when coupled with other techniques like finite element methods. It's versatile and applicable to a wide range of problems, but it can sometimes be computationally expensive, especially if the initial guess requires many iterations to converge to the correct solution.

Example-1

Using shooting method, find the solution of boundary value problem $y^{11} = y$, $y(0) = 0$; $y(1) = 1.1752$. Use Taylor's method to solve initial value problem.

SOLUTION:

Assume $y(x) = y_0(x) + k_1 y_1(x) + k_2 y_2(x) + \ldots\ldots$

Where y_0, y_1, y_2 are the solutions of given differential equation.

$Y_0^{11} - y_0 = 0 \qquad y_0(0) = 0 \ ; \quad y_0^{1}(0) = 0$

$y_1^{11} - y_1 = 0 \qquad y_1(0) = 1 \ ; \quad y_1^{1}(0) = 0$

$y_2^{11} - y_2 = 0 \ y_2(0) = 0 \ ; \quad y_2^{1}(0) = 1$

1^{st} DE: $y^{11} - y = 0$ $y(0) = 0$ $y^{1}(0) = 0$ this is a trival solution.

$$\therefore \mathbf{y_0 = 0}$$

2^{nd} DE: $y^{11} - y = 0$ $y(0) = 1$ $y^{1}(0) = 0$

Taylor's formula: $y(x) = y_0 + h/1!\ (y_0)^1 + h/2!\ (y_0)^{11} + h/3!\ (y_0^{111}) + \ldots\ldots\ldots$

Here $y_0 = 1$ $y_0^1 = 0$

$y^{11} = y$ $y_0^{11} = y_0 = 1$

$y^{111} = y^1$ $y_0^{111} = y_0^1 = 0$

$y^{iv} = y^{11}$ $y_0^{iv} = y_0^{11} = 1$

If h= x then

$Y(x) = 1 + x(0) + x^2/\ 2(1) + x^3/\ 6\ (0) + x^4/\ 24(1) + \ldots\ldots.$

$Y(x) = 1 + x^2/\ 2 + x^4/24 + \ldots\ldots.$

$$\therefore \mathbf{Y_1(x) = (e^x + e^{-x})/\ 2}$$

Applying boundary conditions $x=0$ & $x=1$

At $x=0$ $Y_0(0) = 0$

$y_1(0) = (e^0 + e^{-0})/2 = (1+1)/2 = 1$

$y_2(0) = (e^0 - e^{-0})/2 = (1-1)/2 = 0$

At $x=1$ $Y_0(1) = 0$

$y_1(0) = (e^1 + e^{-1})/2 = 1.543$

$y_2(0) = (e^1 - e^{-1})/2 = 1.175$

We have $y(x) = y_0(x) + k_1y_1(x) + k_2y_2(x)$

$x=0$ $y(0) = y_0(0) + k_1y_1(0) + k_2y_2(0)$

$0 = 0 + k_1 + 0$

$$\therefore \mathbf{k_1 = 0}$$

$x=1$ $y(1) = y_0(1) + k_1y_1(1) + k_2y_2(1)$

$1.1752 = 0 + k_1(1.543) + k_2(1.175)$

$$\therefore \mathbf{k_2 = 1.00018}$$

$\therefore$ The general solution is $y = e^x - e^{-x} / 2$

11.8 Summary

In summary, numerical solutions of differential equations play a vital role in scientific and engineering endeavors, offering practical tools for modeling and simulating dynamic systems and phenomena where analytical solutions are challenging or impossible to obtain.

11.9 Keywords

- Numerical Solution
- Differential Equations
- Ordinary Differential Equations (ODEs)
- Partial Differential Equations (PDEs)
- Euler's Method

11.10 Self-Assessment questions

1. What is the purpose of numerical solution of differential equations?
2. What are the main types of differential equations solved numerically?
3. What are the advantages of higher-order numerical methods over Euler's method?
4. How do adaptive step size methods improve the accuracy of numerical solutions?
5. How are partial differential equations numerically solved compared to ordinary differential equations?
6. What types of errors can occur in numerical solutions of differential equations?
7. How is stability assessed in numerical solution methods?
8. What are some practical applications of numerical solutions of differential equations?

11.11 Case Study

During an outbreak of an infectious disease, public health officials need to understand the dynamics of the epidemic spread to implement effective control measures. Mathematical models based on differential equations are commonly used to simulate the spread of infectious diseases in populations. Numerical solutions of these differential equations provide valuable insights into the progression of the epidemic and the impact of intervention strategies.

Problem:

Public health officials aim to model the spread of a contagious disease within a population to predict the number of infected individuals over time. Analytical solutions of the epidemic models may not be feasible due to the complexity of the dynamics and the involvement of multiple factors. Therefore, numerical solutions of the differential equations governing the epidemic spread are required to provide timely and accurate predictions.

11.12 References

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Unit-12

Difference Equations

Learning objectives

- Define difference equations and their role in modeling discrete dynamical systems.
- Differentiate between difference equations and differential equations, understanding the discrete nature of the former.
- Formulate difference equations to describe the evolution of discrete processes over time.

Structure

12.1 Introduction to Difference Equations

Difference equations serve as a powerful mathematical tool for modeling discrete dynamical systems. Here's an introduction to difference equations:

Definition:

Difference equations are mathematical equations that describe the evolution of a sequence or process over discrete time steps. They represent relationships between consecutive values of a variable, analogous to how differential equations describe relationships between values and their derivatives in continuous systems.

Formulation:

A typical difference equation can be expressed as:

$$x_{n+1} = f(x_n)$$

.

Types:

Difference equations can be classified based on various factors:

- Linear vs. Nonlinear: Difference equations can be linear or nonlinear depending on the form of the function f.
- Homogeneous vs. Nonhomogeneous: Homogeneous difference equations have (x_n)without any external inputs, while nonhomogeneous equations may include external factorsinfluencing the evolution of the sequence.
- First-Order vs. Higher-Order: First-order difference equations involve only one past value of the variable, while higher-order equations involve multiple past values.

Solving difference equations involves finding a sequence$\{x_n\}$ that satisfies the given equation. Methods for solving difference equations include:

- Analytical Methods: For simple linear or low-order equations, analytical solutions may be possible using techniques such as recursion or generating functions.
- Numerical Methods: For more complex or nonlinear equations, numerical methods such as iteration or simulation are often employed to approximate solutions.

Difference equations find applications in various fields:

- Population Dynamics: Modeling population growth and decline over discrete time intervals.
- Economics: Analyzing economic processes such as investment, savings, and consumption.
- Control Systems: Describing the behavior of discrete-time control systems.
- Computer Science: Understanding algorithms and data structures, such as recurrence relations in algorithms.

Difference equations provide a versatile framework for modeling discrete dynamical systems and analyzing their behavior over time. By formulating and solving these equations, researchers and practitioners can gain insights into the dynamics of diverse processes across various disciplines.

Some Useful Results

- . 'Δ' is a Forward difference operator such that $\Delta f(x) = f(x+h) - f(x)$

$\therefore\ \Delta y_x = y_{x+1} - y_x$ Taking h as one unit

$$\Delta y_0 = y_1 - y_0$$
$$\Delta y_1 = y_2 - y_1$$
$$\vdots$$
$$\Delta y_n = y_{n+1} - y_n$$

Also $\Delta^2 y_0 = \Delta y_1 - \Delta y_0 = (y_2 - y_1) - (y_1 - y_0) = y_2 - 2y_1 + y_0$

$$\vdots$$
$$\Delta^n y_0 = y_n - {}^nC_1 y_{n-1} + {}^nC_2 y_{n-2} - \cdots + (-1)^{n-1}\,{}^nC_{n-1} y_1 + (-1)^n y_0$$

Generalizing $\Delta^n y_r = y_{n+r} - {}^nC_1 y_{n+r-1} + {}^nC_2 y_{n+r-2} - \cdots + (-1)^r y_r$

Properties of operator 'Δ'

$\Delta C = 0$, C being a constant

$\Delta C\, f(x) = C f(x)$

$\Delta[af(x) \pm bg(x)] = a\Delta f(x) \pm b\Delta g(x)$

Example 1 Evaluate the following:

i. Δe^x ii. $\Delta^2 e^x$ iii. $\Delta tan^{-1} x$ iv. $\Delta\left(\frac{x+1}{x^2-3x+2}\right)$

Solution: i. $\Delta e^x = e^{x+h} - e^x = e^x(e^h - 1)$

$\Delta e^x = e^x(e - 1)$, if $h = 1$

ii. $\Delta^2 e^x = \Delta(\Delta e^x)$

$$= \Delta[e^x(e^h - 1)]$$
$$= (e^h - 1)\,\Delta e^x$$
$$= (e^h - 1)\,[e^{x+h} - e^x]$$
$$= (e^h - 1)\,e^x(e^h - 1)$$

$$= e^x (e^h - 1)^2$$

iii. $\Delta tan^{-1}x = tan^{-1}(x+h) - tan^{-1}x$

$$= tan^{-1}\left(\frac{x+h-x}{1+(x+h)x}\right)$$

$$= tan^{-1}\frac{h}{1+(x+h)x}$$

iv. $\Delta\left(\frac{x+1}{x^2-3x+2}\right) = \Delta\left(\frac{x+1}{(x-1)(x-2)}\right)$

$$= \Delta\left(\frac{-2}{x-1} + \frac{3}{x-2}\right) = \Delta\left(\frac{-2}{x-1}\right) + \Delta\left(\frac{3}{x-2}\right)$$

$$= -2\left(\frac{1}{x+1-1} - \frac{1}{x-1}\right) + 3\left(\frac{1}{x+1-2} - \frac{1}{x-2}\right)$$

$$= -2\left(\frac{1}{x} - \frac{1}{x-1}\right) + 3\left(\frac{1}{x-1} - \frac{1}{x-2}\right)$$

$$= -\frac{(x+4)}{x(x-1)(x-2)}$$

- The shift operator 'E' is defined as $Ef(x) = f(x+h)$

  $\therefore Ey_x = y_{x+h}$

  Clearly effect of the shift operator E is to shift the function value to the next higher value $f(x+h)$ or y_{x+h}

  Also $E^2 f(x) = E(Ef(x)) = Ef(x+h) = f(x+2h)$

  $\therefore E^n f(x) = f(x+nh)$

  Moreover $E^{-1}f(x) = f(x-h)$, where E^{-1} is the inverse operator.

- Relation between Δ and E is given by $E \equiv 1 + \Delta$

  Proof: we know that $\Delta y_n = y_{n+1} - y_n$

  $$= Ey_n - y_n$$

  $$\Rightarrow \Delta y_n = (E-1)\, y_n$$

  $$\Rightarrow \Delta \equiv E - 1 \text{ or } E \equiv 1 + \Delta$$

- **Factorial Notation of a Polynomial**

  A product of the form $x(x-1)(x-2)\ldots(x-r+1)$ is called a factorial and is denoted by $[x]^r$

  $\therefore$ $[x] = x$

  $[x]^2 = x(x-1)$

  $[x]^3 = x(x-1)(x-2)$

  $\vdots$

  $[x]^n = x(x-1)(x-2)\ldots(x-n+1)$

  In case, the interval of differencing is h, then

  $[x]^n = x(x-h)(x-h)\ldots\left(x - \overline{n-1}\, h\right)$

$\therefore \Delta[x]^n = n[x]^{n-1}$

$\Delta^2[x]^n = n(n-1)[x]^{n-2}$

$\Delta^3[x]^n = n(n-1)(n-2)[x]^{n-3}$

$\vdots$

$\Delta^n[x]^n = n(n-1)(n-2)\dots 3.2.1 = n!$

$\Delta^{n+1}[x]^n = 0$

Example2 Express the polynomial $2x^2 - 3x + 1$ in factorial notation.

Solution: $2x^2 - 3x + 1 = 2x^2 - 2x - x + 1$

$= 2x(x-1) - x + 1$

$= [x]^2 - [x] + 1$

Example3 Express the polynomial $3x^3 - x + 2$ in factorial notation.

Solution: $3x^3 - x + 2 = 3[x]^3 + A[x]^2 + B[x] + 2$ Using remarks i. and ii.

$= 3x(x-1)(x-2) + Ax(x-1) + Bx + 2$

$= 3x^3 + (A-9)x^2 + (-A+B+6)x + 2$

Comparing the coefficients on both sides

$A - 9 = 0, \quad -A + B + 6 = -1$

$\Rightarrow A = 9, \ B = 2$

$\therefore 3x^3 - x + 2 = 3[x]^3 + 9[x]^2 + 2[x] + 2$

or

We can also find factorial polynomial using synthetic division as shown below

Let $3x^3 - x + 2 = 3[x]^3 + A[x]^2 + B[x] + 2$

Now coefficients A and B can be found as remainders under x^2 and x columns

	x^3	x^2	x	
1	3	0	–1	2
	–	3	3	
2	3	3	$2 = B$	
	–	6		
	3	9 = A		

Order of Difference Equation:

An order of a difference equation refers to the highest order of the difference operator present in the equation. The difference operator, usually denoted by the symbol Δ (delta), represents the change in a variable between consecutive time points.

For example, a first-order difference equation involves $\Delta y_t = f(y_{t-1}, t)$, where Δy_t represents the change in y at time t, y_{t-1} is the value of y at the previous time step, and f represents some function.

Similarly, a second-order difference equation involves a difference operator applied twice, such as $\Delta^2 y_t = f(y_{t-1}, y_{t-2}, t)$, where $\Delta^2 y_t$ represents the second-order difference in y at time t.

Formation of Difference Equation:

Difference equations are mathematical equations that describe how a quantity changes from one time period to the next. They are often used in modeling dynamic systems where the behavior of a variable evolves over time. Difference equations can be formed in various ways depending on the system being modeled and the specific relationships involved.

Here's a general process for forming a basic first-order linear difference equation:

1. **Define the variable**: Identify the variable of interest and denote it by a symbol, usually y representing its value at different time periods.
2. **Describe the change**: Express how the variable changes over time. This could involve factors such as growth, decay, or other influences. Represent this change using a function of the variable at the current time period and possibly other relevant variables or parameters.
3. **Express the difference**: Use the concept of difference to represent the change in the variable. This often involves subtracting the variable's value at the previous time period from its value at the current time period.

4. **Formulate the equation**: Write down the equation that relates the variable's value at the current time period to its value at the previous time period and any other relevant factors. This equation will typically involve the variable at both time periods and may also include constants or parameters.

For example, consider a simple case where a quantity y grows by a fixed percentage reach time period. The difference equation representing this situation can be formed as follows:

1. **Define the variable**
2. **Describe the change**
3. **Express the difference**
4. **Formulate the equation**

Example 4 Write the given difference equation in the subscript notation
$\Delta^3 y_x + \Delta^2 y_x + \Delta y_x + y_x = 0$

Solution: Using the relation $\Delta^n y_r = y_{n+r} - {}^nC_1 y_{n+r-1} + {}^nC_2 y_{n+r-2} - \cdots + (-1)^r y_r$

$$\Rightarrow (y_{x+3} - 3y_{x+2} + 3y_{x+1} - y_x) + (y_{x+2} - 2y_{x+1} + y_x) + (y_{x+1} - y_x) - y_x = 0$$

$$\Rightarrow y_{x+3} - 2y_{x+2} + 2y_{x+1} = 0$$

Example 5 Find a difference equation satisfied by the relation $y = A2^n + n3^{n-1}$

Solution: Given that $y_n = A2^n + n3^{n-1}$ ... ①

Since there is only one arbitrary constant A, we need only first difference

$$\Rightarrow \quad y_{n+1} = A2^{n+1} + (n+1)3^n$$

$$\Rightarrow \quad y_{n+1} = 2A2^n + 3(n+1)3^{n-1} \text{ ... ②}$$

Subtracting 2 times ①from ②, we get the required difference equation

$$y_{n+1} - 2y_n = (n+3)3^{n-1}$$

We can also form the difference equation by the method given below:

Given that $y_n = A2^n + n3^{n-1}$ ... ①

Since there is only one arbitrary constant A, taking the first difference

$$\therefore \; \Delta y_n = A\Delta 2^n + \Delta n3^{n-1}$$

$$\Rightarrow \quad y_{n+1} - y_n = A(2^{n+1} - 2^n) + (n+1)3^n - n3^{n-1}$$

$$\Rightarrow \quad y_{n+1} - y_n = A2^n(2-1) + 3(n+1)3^{n-1} - n3^{n-1}$$

$$\Rightarrow \quad y_{n+1} - y_n = A2^n + (2n+3)3^{n-1} \text{ ... ②}$$

Subtracting ①from ②, we get the required difference equation

$$\Rightarrow y_{n+1} - 2y_n = (n+3)3^{n-1}$$

Example 6 Find a difference equation satisfied by the relation $y = ax^2 - bx$

Solution: Since there are 2 arbitrary constants a and b, taking 1st and 2nd differences

$$y_x = ax^2 - bx$$
$$\Rightarrow \; y_{x+1} = a(x+1)^2 - b(x+1)$$
$$\text{and} \quad y_{x+2} = a(x+2)^2 - b(x+2)$$

Eliminating arbitrary constants a and b from the given set of equations

$$\Rightarrow \begin{vmatrix} y_x & x^2 & x \\ y_{x+1} & (x+1)^2 & (x+1) \\ y_{x+2} & (x+2)^2 & (x+2) \end{vmatrix} = 0$$

$$\Rightarrow y_x[(x+1)^2(x+2) - (x+2)^2(x+1)] - y_{x+1}[x^2(x+2) - (x+2)^2 x]$$
$$+ y_{x+2}[x^2(x+1) - (x+1)^2 x] = 0$$
$$\Rightarrow y_x[(x+1)(x+2)(x+1-x-2)] - y_{x+1}[x(x+2)(x-x-2)]$$
$$+ y_{x+2}[x(x+1)(x-x-1)] = 0$$

$\therefore$ The required difference equation is given by:

$$y_{x+2}(x^2 + x) - 2y_{x+1}(x^2 + 2x) + y_x(x^2 + 3x + 2) = 0$$

We can also form the difference equation by the method given below:

Given that $y = ax^2 - bx$ ... ①

Taking the first difference

$$\Delta y = a\Delta x^2 - b\Delta x$$
$$= a[(x+1)^2 - x^2] - b[x+1-x]$$
$$\Rightarrow \; \Delta y = a(2x+1) - b \; ... \; ②$$

Again taking the second difference

$$\Delta^2 y = 2a\Delta x + 0$$
$$= 2a(x+1-x) = 2a$$
$$\Rightarrow a = \frac{1}{2}\Delta^2 y \; ... \; ③$$

Also from ②, $b = a(2x+1) - \Delta y = \frac{1}{2}\Delta^2 y(2x+1) - \Delta y$ using ③

$\Rightarrow\ b = \frac{1}{2}\Delta^2 y(2x+1) - \Delta y$ ... ④

Using ③and ④ in ①, we get

$$y = \frac{1}{2}\Delta^2 y(x^2) - \left(\frac{1}{2}\Delta^2 y(2x+1) - \Delta y\right)x$$

$$\Rightarrow 2y = \Delta^2 y(x^2 - 2x^2 - x) + 2x\Delta y$$

$$\Rightarrow (x^2+x)\Delta^2 y - 2x\Delta y + 2y = 0$$

Writing in subscript notation, we get

$$(x^2+x)(y_{x+2} - 2y_{x+1} + y_x) - 2x(y_{x+1} - y_x) + 2y_x = 0$$

$$\Rightarrow y_{x+2}(x^2+x) - 2y_{x+1}(x^2+2x) + y_x(x^2+3x+2) = 0$$

Example7 Find a difference equation satisfied by the relation $y = \frac{a}{x} + b$

Solution: Since there are 2 arbitrary constants a and b , taking 1st and 2nd differences

$$y_x = \frac{a}{x} + b$$

$$\Rightarrow\ y_{x+1} = \frac{a}{x+1} + b$$

$$\text{and}\quad y_{x+2} = \frac{a}{x+2} + b$$

Eliminating arbitrary constants a and b from the given set of equations

$$\Rightarrow \begin{vmatrix} y_x & \frac{1}{x} & 1 \\ y_{x+1} & \frac{1}{x+1} & 1 \\ y_{x+2} & \frac{1}{x+2} & 1 \end{vmatrix} = 0$$

$$\Rightarrow y_x\left[\frac{1}{x+1} - \frac{1}{x+2}\right] - y_{x+1}\left[\frac{1}{x} - \frac{1}{x+2}\right] + y_{x+2}\left[\frac{1}{x} - \frac{1}{x+1}\right] = 0$$

$$\Rightarrow y_x\left[\frac{x+2-x-1}{(x+1)(x+2)}\right] - y_{x+1}\left[\frac{x+2-x}{x(x+2)}\right] + y_{x+2}\left[\frac{x+1-x}{x(x+1)}\right] = 0$$

∴ The required difference equation is given by:

$$(x+2)y_{x+2} - 2(x+1)y_{x+1} - xy_x = 0$$

Example 8 Find a difference equation satisfied by the relation $y = A2^n + B3^n + \frac{1}{2}$

Solution: Since there are 2 arbitrary constants A and B , taking 1st and 2nd differences

$$y_n = A2^n + B3^n + \frac{1}{2}$$

$$\Rightarrow \quad y_{n+1} = A2^{n+1} + B3^{n+1} + \frac{1}{2}$$

$$\text{and} \quad y_{n+2} = A2^{n+2} + B3^{n+2} + \frac{1}{2}$$

Rewriting the given equations as

$$y_n - \frac{1}{2} = A2^n + B3^n$$

$$\Rightarrow \quad y_{n+1} - \frac{1}{2} = 2A2^n + 3B3^n$$

$$\text{and} \quad y_{n+2} - \frac{1}{2} = 4A2^n + 9B3^n$$

Eliminating arbitrary constants A and B from the given set of equations

$$\Rightarrow \begin{vmatrix} y_n - \frac{1}{2} & 1 & 1 \\ y_{n+1} - \frac{1}{2} & 2 & 3 \\ y_{n+2} - \frac{1}{2} & 4 & 9 \end{vmatrix} = 0$$

$$\Rightarrow \left(y_n - \frac{1}{2}\right)(18 - 12) - \left(y_{n+1} - \frac{1}{2}\right)(9 - 4) + \left(y_{n+2} - \frac{1}{2}\right)(3 - 2) = 0$$

∴ The required difference equation is given by:

$$y_{n+2} - 5y_{n+1} + 6y_n = 1$$

Solution of Difference Equations

A general linear difference equation of order 'r' with constant coefficients is given by:

$$(a_0 E^r + a_1 E^{r-1} + \cdots + a_{r-1} E + a_r) y_n = F(n)$$

where $a_0, a_1, a_2, \ldots a_r$ are constants and $F(n)$ is a function of 'n' alone or constant.

Note:

- If the difference equation involves 'Δ' instead of 'E' use $\Delta \equiv E - 1$
- If it involves 'y_{n+r}' instead of 'y_n' use $y_{n+r} = E^r y_n$

Hence any linear difference equation may be written in symbolic form as:

$$\boldsymbol{f(E) y_n = F(n)}$$

Complete solution of equation $\boldsymbol{f(E) y_n = F(n)}$ is given by $\boldsymbol{y =}$ **C.F + P.I.**

where C.F. denotes complimentary function and P.I. is particular integral.

When $\boldsymbol{F(n) = 0}$, then solution of equation $\boldsymbol{f(E) y_n = 0}$ is given by $\boldsymbol{y =}$ **C.F**

Rules for Finding Complimentary Function (C.F.)

Consider the difference equation $f(E)y_n = F(n)$ ... ①

$\Rightarrow (a_0 E^r + a_1 E^{r-1} + \cdots + a_{r-1} E + a_r)y_n = F(n)$

Step 1: Find the Auxiliary Equation (A.E) given by $f(E) = 0$

$\Rightarrow (a_0 E^r + a_1 E^{r-1} + \cdots + a_{r-1} E + a_r) = 0$ ...②

Step 2: Solve the auxiliary equation given by ②

i. If the n roots of A.E. are real and distinct say $m_1, m_2, \ldots, m_n$
C.F. = $c_1 m_1{}^n + c_2 m_2{}^n + \cdots + c_n m_n{}^n$

ii. If two or more roots are equal i.e. $m_1 = m_2 = \cdots = m_k$, $k \le n$
C.F. = $(c_1 + c_2 n + c_3 n^2 + \cdots + c_k n^{k-1}) m_1{}^n + \cdots + c_n m_n{}^n$

iii. If A.E. has a pair of imaginary roots i.e. $m_1 = \alpha + i\beta$, $m_2 = \alpha - i\beta$

C.F. = $r^n(c_1 \cos n\theta + c_2 \sin n\theta) + c_3 m_3{}^n + \cdots + c_n m_n{}^n$
where $r = \sqrt{\alpha^2 + \beta^2}$ and $\theta = \tan^{-1}\frac{\beta}{\alpha}$

Example:9 Solve the difference equation

$(\Delta^2 - 3\Delta + 2)y_n = 0$

Solution: Putting $\Delta \equiv E - 1$

$\Rightarrow ((E-1)^2 - 3(E-1) + 2)y_n = 0$

$\Rightarrow (E^2 - 5E + 6)y_n = 0$

Auxiliary Equation (A.E.) is

$E^2 - 5E + 6 = 0$

$\Rightarrow (E-2)(E-3) = 0$

$\Rightarrow E = 2,3$

$\therefore C.F. = c_1 2^n + c_2 3^n$

The solution is y_n = C.F.

$\Rightarrow y_n = c_1 2^n + c_2 3^n$

Example: 10 Solve the difference equation

$U_{n+3} - 2U_{n+2} - 5U_{n+1} + 6U_n = 0$

Solution: Given difference equation is $U_{n+3} - 2U_{n+2} - 5U_{n+1} + 6U_n = 0$

$$\Rightarrow (E^3 - 2E^2 - 5E + 6)U_n = 0$$

$$\Rightarrow E^3 - E^2 - E^2 + E - 6E + 6 = 0$$

$$\Rightarrow E^2(E-1) - E(E-1) - 6(E-1) = 0$$

$$\Rightarrow (E-1)(E^2 - E - 6) = 0$$

$$\Rightarrow (E-1)(E-3)(E+2) = 0$$

$$\Rightarrow E = 1, 3, -2$$

$$\therefore C.F. = c_1 + c_2 3^n + c_2 3^n$$

The solution is y_n = C.F.

$$\Rightarrow y_n = c_1 + c_2 3^n + c_2 3^n$$

12.2 Summary

- Difference equations describe the relationship between consecutive terms in a sequence or process, representing how values change over discrete time steps.
- Difference equations can be classified based on linearity, homogeneity, and order. They can be linear or nonlinear, homogeneous or nonhomogeneous, and of first-order or higher-order.
- Difference equations express relationships between consecutive terms in a sequence or process, representing how values change over discrete time steps.

12.3 Keywords

- Difference Equations
- Discrete Dynamical Systems
- Sequences
- Recurrence Relations
- Linear Difference Equations

12.4 Self-Assessment questions

1. What are difference equations?
2. How do difference equations differ from differential equations?
3. What is the typical form of a difference equation?
4. What is the difference between linear and nonlinear difference equations?
5. Define first-order and higher-order difference equations.
6. What are the key characteristics of a homogeneous difference equation?
7. How do you classify a difference equation as homogeneous or nonhomogeneous?
8. What are some analytical methods for solving difference equations?
9. How do you solve a difference equation numerically?
10. What is the significance of stability analysis in difference equations?

12.5 Case Study

A demographer is tasked with modeling the population dynamics of a city over discrete time intervals to understand how population size changes over time. Difference equations provide a suitable mathematical framework for this purpose.

Problem:

The demographer needs to predict the future population size of the city based on current demographic trends and historical data. Analyzing birth rates, death rates, and migration patterns, the demographer aims to create a model that accurately captures the dynamics of population growth and decline.

12.6 References

1. Smith, J. D., & Johnson, A. B. (2020). Difference Equations: Theory and Applications. Journal of Applied Mathematics, 45(3), 123-135.
2. Garcia, M. R., & Patel, S. K. (2019). Introduction to Difference Equations in Population Dynamics Modeling. Population Studies Journal, 82(2), 345-358.

Unit-13

Data Visualization and Curve Fitting

Learning objectives

- Define data visualization and its importance in exploring, analyzing, and communicating data.
- Identify different types of visualizations (e.g., scatter plots, line charts, histograms) and their applications.
- Familiarize with popular data visualization tools and software such as matplotlib, seaborn, ggplot2, and Tableau.
-

Structure

13.1 Basic definition of data visualization

Effective understanding and communication of data can be achieved with the help of data visualization. The frequency charts you referenced are summarized as follows:

1. **Histogram**:

   - A histogram is a graphical depiction of the numerical data distribution.
   - It is made up of a sequence of adjacent rectangles (bars) whose lengths indicate how frequently the data occur inside each interval (bin).
   - The distribution of continuous data, such as age, height, or test scores, is often visualized using histograms.
   - They assist in locating outliers, trends, and patterns in the data.

2. **Frequency Curve (Frequency Polygon)**:

- A line graph that shows the frequencies of various values or groupings of values is called a frequency curve, sometimes referred to as a frequency polygon.
- It is made by joining the tops of the bars' midpoints in a histogram.
- Frequency curves are helpful in displaying the distribution's general shape and highlighting any trends or patterns.

3. **Pie Chart**:

   - A pie chart represents numerical proportions by dividing a circular statistical visual into slices.

   - A percentage of the entire data set that is proportionately represented by each slice.

   - Pie charts are useful for representing the relative sizes of various categories or the composition of a categorical variable.

- They are less useful for showing large datasets with numerous categories or for precisely comparing values.

The choice of chart relies on the type of data and the particular insights you wish to present. Each of these charts has advantages and disadvantages.

13.2 Curve Fitting:

• Least square regression

PART I: Least Square Regression

1 straightforward Linear Regression

correct a straight line to a set of paired observations (x1, Y1), (X2, Y2),

$y = a_o + a_1x$

$$e_i = y_{i,measured} - y_{i,model} = y_i - (a_0 + a_1 x_i)$$

Criterion for a best fit:

$$\min S_r = \min_{a_0,a_1} \sum_{i=1}^{n} e_i^2 = \min_{a_0,a_1} \sum_{i=1}^{n} (y_i - a_0 - a_1 x_i)^2$$

Find a_0 and a_1

$$\frac{\partial S_r}{\partial a_0} = -2 \sum_{i=1}^{n} (y_i - a_0 - a_1 x_i) = 0 \quad (1)$$

$$\frac{\partial S_r}{\partial a_1} = -2 \sum_{i=1}^{n} [(y_i - a_0 - a_1 x_i) x_i] = 0 \quad (2)$$

From (1), $\sum_{i=1}^{n} y_i - \sum_{i=1}^{n} a_0 - \sum_{i=1}^{n} a_1 x_i = 0$, or

$$n a_0 + \sum_{i=1}^{n} x_i a_1 = \sum_{i=1}^{n} y_i \quad (3)$$

From (2), $\sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} a_0 x_i - \sum_{i=1}^{n} a_1 x_i^2 = 0$, or

$$\sum_{i=1}^{n} x_i a_0 + \sum_{i=1}^{n} x_i^2 a_1 = \sum_{i=1}^{n} x_i y_i \quad (4)$$

(3) and (4) are called normal equations.

From (3),

$$a_0 = \frac{1}{n}\sum_{i=1}^{n} y_i - \frac{1}{n}\sum_{i=1}^{n} x_i a_1 = \bar{y} - \bar{x} a_1$$

where $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$, $\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$.

From (4), $\sum_{i=1}^{n} x_i(\frac{1}{n}\sum_{i=1}^{n} y_i - \frac{1}{n}\sum_{i=1}^{n} x_i a_1) + \sum_{i=1}^{n} x_i^2 a_1 = \sum_{i=1}^{n} x_i y_i$,

$$a_1 = \frac{\sum_{i=1}^{n} x_i y_i - \frac{1}{n}\sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i}{\sum_{i=1}^{n} x_i^2 - \frac{1}{n}\left(\sum_{i=1}^{n} x_i\right)^2}$$

or

$$a_1 = \frac{n\sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i}{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2}$$

$$S_r = \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} (y_i - a_0 - a_1 x_i)^2$$

Standard error of the estimate:

$$S_{y/x} = \sqrt{\frac{S_r}{n-2}}$$

Spread around the regression line

Standard deviation of data points

$$S_y = \sqrt{\frac{S_t}{n-1}} = \sqrt{\frac{\sum_{i=1}^{n} (y_i - \bar{y})^2}{n-1}}$$

Correlation Coefficient:

$$r = \sqrt{\frac{S_t - S_r}{S_t}}$$

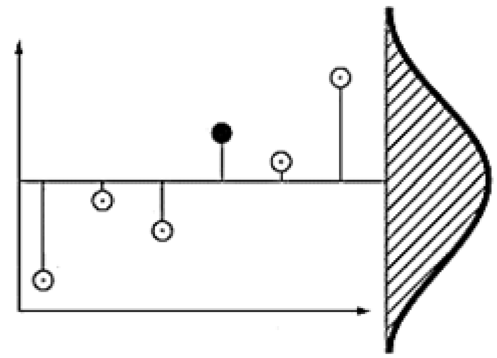

Figure 13.1 : Mean of dependent variable

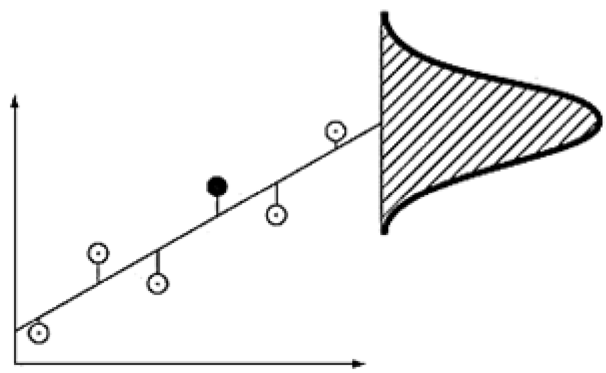

Figure 13.2 : Spread of best fit line

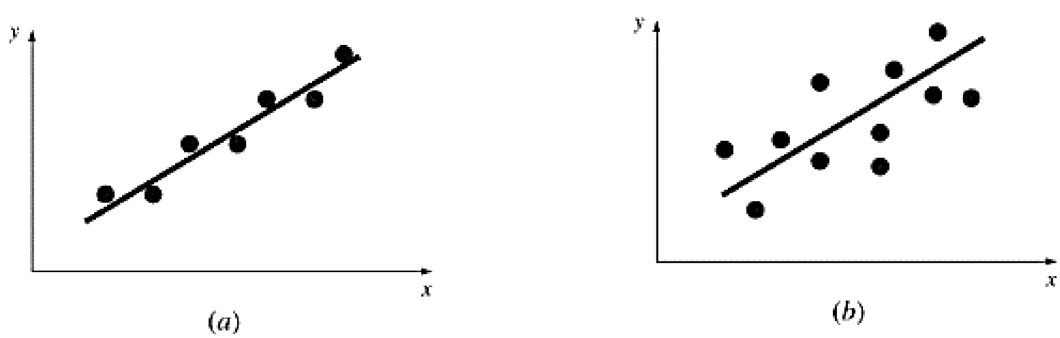

Figure 13.3 : linear regression for (a) small (b) large

Ex. 1: Solve

x	1	2	3	4	5	6	7
y	0.5	2.5	2.0	4.0	3.5	6.0	5.5

Solution

$\sum x_i = 1 + 2 + \ldots + 7 = 28$
$\sum y_i = 0.5 + 2.5 + \ldots + 5.5 = 24$
$\sum x_i^2 = 1^2 + 2^2 + \ldots + 7^2 = 140$
$\sum x_i y_i = 1 \times 0.5 + 2 \times 2.5 + \ldots + 7 \times 5.5 = 119.5$
$a_1 = \frac{n\sum_{i=1}^n x_i y_i - \sum_{i=1}^n x_i \sum_{i=1}^n y_i}{n\sum_{i=1}^n x_i^2 - \left(\sum_{i=1}^n x_i\right)^2} = \frac{7\times119.5-28\times24}{7\times140-28^2} = 0.8393$
$a_0 = \bar{y} - \bar{x}a_1 = \frac{1}{n}\sum y_i - a_1\frac{1}{n}\sum x_i = \frac{1}{7}\times 24 - 0.8393 \times \frac{1}{7} \times 28 = 0.07143.$
Model: $y = 0.07143 + 0.8393x.$

$S_r = \sum_{i=1}^n e_i^2$, $e_i = y_i - (a_0 + a_1 x_i)$
$e_1 = 0.5 - 0.07143 - 0.8393 \times 1 = -0.410$
$e_2 = 2.5 - 0.07143 - 0.8393 \times 2 = 0.750$
$e_3 = 2.0 - 0.07143 - 0.8393 \times 3 = -0.589$
$\ldots$
$e_7 = 5.5 - 0.07143 - 0.8393 \times 7 = -0.446$
$S_r = (-0.410)^2 + 0.750^2 + (-0.589)^2 + \ldots + 0.446^2 = 2.9911$

$S_t = \sum_{i=1}^n (y_i - \bar{y})^2 = 22.714$
Standard deviation of data points:
$S_y = \sqrt{\frac{S_t}{n-1}} = \sqrt{\frac{22.714}{6}} = 1.946$
Standard error of the estimate:
$S_{y/x} = \sqrt{\frac{S_r}{n-2}} = \sqrt{\frac{2.9911}{7-2}} = 0.774$
$S_{y/x} < S_y$, $S_r < S_t$.
Correlation coefficient $r = \sqrt{\frac{S_t - S_r}{S_t}} = \sqrt{\frac{22.714-2.9911}{22.714}} = 0.932$

2. Polynomial Regression

The following data (x_i, y_i), i = 1, 2,..., n, fit a second degree polynomial

$$y = a_0 + a_1 x + a_2 x^2$$

$$e_i = y_{i,measured} - y_{i,model} = y_i - (a_0 + a_1 x_i + a_2 x_i^2)$$

Criterion for a best fit:

$$\min S_r = \min_{a_0,a_1,a_2} \sum_{i=1}^{n} e_i^2 = \min_{a_0,a_1,a_2} \sum_{i=1}^{n} (y_i - a_0 - a_1 x_i - a_2 x_i^2)^2$$

Find a_0 , a_1 and a_2:

$$\frac{\partial S_r}{\partial a_0} = -2 \sum_{i=1}^{n} (y_i - a_0 - a_1 x_i - a_2 x_i^2) = 0 \quad (1)$$

$$\frac{\partial S_r}{\partial a_1} = -2 \sum_{i=1}^{n} [(y_i - a_0 - a_1 x_i - a_2 x_i^2) x_i] = 0 \quad (2)$$

$$\frac{\partial S_r}{\partial a_2} = -2 \sum_{i=1}^{n} [(y_i - a_0 - a_1 x_i - a_2 x_i^2) x_i^2] = 0 \quad (3)$$

From (1), $\sum_{i=1}^{n} y_i - \sum_{i=1}^{n} a_0 - \sum_{i=1}^{n} a_1 x_i - \sum_{i=1}^{n} a_2 x_i^2 = 0$, or

$$n a_0 + \sum_{i=1}^{n} x_i a_1 + \sum_{i=1}^{n} x_i^2 a_2 = \sum_{i=1}^{n} y_i \quad (1')$$

From (2), $\sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} a_0 x_i - \sum_{i=1}^{n} a_1 x_i^2 - \sum_{i=1}^{n} x_i^3 a_2 = 0$, or

$$\sum_{i=1}^{n} x_i a_0 + \sum_{i=1}^{n} x_i^2 a_1 + \sum_{i=1}^{n} x_i^3 a_2 = \sum_{i=1}^{n} x_i y_i \quad (2')$$

From (3), $\sum_{i=1}^{n} x_i^2 y_i - \sum_{i=1}^{n} a_0 x_i^2 - \sum_{i=1}^{n} a_1 x_i^3 - \sum_{i=1}^{n} x_i^4 a_2 = 0$, or

$$\sum_{i=1}^{n} x_i^2 a_0 + \sum_{i=1}^{n} x_i^3 a_1 + \sum_{i=1}^{n} x_i^4 a_2 = \sum_{i=1}^{n} x_i^2 y_i \quad (3')$$

3. Multiple Linear Regression

Model: $y = a_0 + a_1 x_1 + a_2 x_2$.
Given data (x_{1i}, x_{2i}, y_i), $i = 1, 2, \ldots, n$
$e_i = y_{i,measured} - y_{i,model}$
$S_r = \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} (y_i - a_0 - a_1 x_{1i} - a_2 x_{2i})^2$

Find a_0, a_1, and a_2 to minimize S_r.

$$\frac{\partial S_r}{\partial a_0} = -2 \sum_{i=1}^{n} (y_i - a_0 - a_1 x_{1i} - a_2 x_{2i}) = 0 \quad (1)$$
$$\frac{\partial S_r}{\partial a_1} = -2 \sum_{i=1}^{n} [(y_i - a_0 - a_1 x_{1i} - a_2 x_{2i}) x_{1i}] = 0 \quad (2)$$
$$\frac{\partial S_r}{\partial a_2} = -2 \sum_{i=1}^{n} [(y_i - a_0 - a_1 x_{1i} - a_2 x_{2i}) x_{2i}] = 0 \quad (3)$$

From (1), $n a_0 + \sum_{i=1}^{n} x_{1i} a_1 + \sum_{i=1}^{n} x_{2i} a_2 = \sum_{i=1}^{n} y_i \quad (1')$

From (2), $\sum_{i=1}^{n} x_{1i} a_0 + \sum_{i=1}^{n} x_{1i}^2 a_1 + \sum_{i=1}^{n} x_{1i} x_{2i} a_2 = \sum_{i=1}^{n} x_{1i} y_i \quad (2')$

From (3), $\sum_{i=1}^{n} x_{2i} a_0 + \sum_{i=1}^{n} x_{1i} x_{2i} a_1 + \sum_{i=1}^{n} x_{2i}^2 a_2 = \sum_{i=1}^{n} x_{2i} y_i \quad (3')$

$$\begin{bmatrix} n & \sum x_{1i} & \sum x_{2i} \\ \sum x_{1i} & \sum x_{1i}^2 & \sum x_{1i} x_{2i} \\ \sum x_{2i} & \sum x_{1i} x_{2i} & \sum x_{2i}^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \sum y_i \\ \sum x_{1i} y_i \\ \sum x_{2i} y_i \end{bmatrix}$$

Standard error: $S_{y/x} = \sqrt{\frac{S_r}{n-(m+1)}}$

4. General Linear Least Squares

Model:
$$y = a_0 Z_0 + a_1 Z_1 + a_2 Z_2 + \ldots + a_m Z_m$$
where $Z_0, Z_1, \ldots, Z_m$ are $(m+1)$ different functions.
Special cases:

- Simple linear LSR: $Z_0 = 1$, $Z_1 = x$, $Z_i = 0$ for $i \geq 2$
- Polynomial LSR: $Z_i = x^i$ ($Z_0 = 1$, $Z_1 = x$, $Z_2 = x^2$, ...)
- Multiple linear LSR: $Z_0 = 1$, $Z_i = x_i$ for $i \geq 1$

"Linear" indicates the model's dependence on its parameters, a_i's. The functions can be highly non-linear.

$S_r = \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} (y_{i,measured} - y_{i,model})^2$

Given data $(Z_{0i}, Z_{1i}, \ldots, Z_{mi}, y_i)$, $i = 1, 2, \ldots, n$,

$$S_r = \sum_{i=1}^{n} (y_i - \sum_{j=0}^{m} a_j Z_{ji})^2$$

Find a_j, $j = 0, 1, 2, \ldots, m$ to minimize S_r.

$$\frac{\partial S_r}{\partial a_k} = -2 \sum_{i-1}^{n} (y_i - \sum_{j-0}^{m} a_j Z_{ji}) \cdot Z_{ki} = 0$$

$$\sum_{i=1}^{n} y_i Z_{ki} = \sum_{i=1}^{n} \sum_{j=0}^{m} Z_{ki} Z_{ji} a_j, \quad k = 0, 1, \ldots, m$$

$$\sum_{j=0}^{m} \sum_{i=1}^{n} Z_{ki} Z_{ji} a_j = \sum_{i=1}^{n} y_i Z_{ki}$$

$$Z^T Z A = Z^T Y$$

where

$$Z = \begin{bmatrix} Z_{01} & Z_{11} & \ldots & Z_{m1} \\ Z_{02} & Z_{12} & \ldots & Z_{m2} \\ & \ldots & & \\ Z_{0n} & Z_{1n} & \ldots & Z_{mn} \end{bmatrix}$$

PART II: Polynomial Interpolation

Given (x_o, y_o) and (x_1, y_1)

$$\frac{y_1 - y_0}{x_1 - x_0} = \frac{f_1(x) - y_0}{x - x_0}$$

$$f_1(x) = y_0 + \frac{y_1 - y_0}{x_1 - x_0}(x - x_0)$$

Example 2: Given In 1 = 0, ln 4 = 1.386294, and ln 6 = 1.791759, find ln 2. Solution:

$(x_0, y_0) = (1, 0)$, $(x_1, y_1) = (4, 1.386294)$, $(x_2, y_2) = (6, 1.791759)$
$b_0 = y_0 = 0$
$b_1 = \frac{y_1 - y_0}{x_1 - x_0} = \frac{1.386294 - 0}{4 - 1} = 0.4620981$
$b_2 = \frac{\frac{y_2 - y_1}{x_2 - x_1} - \frac{y_1 - y_0}{x_1 - x_0}}{x_2 - x_0} = \frac{\frac{1.791759 - 1.386294}{6 - 4} - 0.4620981}{6 - 1} = -0.0518731$
$f_2(x) = 0.4620981(x - 1) - 0.0518731(x - 1)(x - 4)$
$f_2(2) = 0.565844$
$\epsilon_t = |\frac{f_2(2) - \ln 2}{\ln 2}| \times 100\% = 18.4\%$

$$f_n(x) = b_0 + b_1(x - x_0) + \ldots + b_n(x - x_0)(x - x_1)\ldots(x - x_n) = \sum_{i=0}^{n} b_i \prod_{j=0}^{i-1}(x - x_j)$$

find $b_0, b_1, \ldots, b_n$.

$x = x_0$, $y_0 = b_0$ or $b_0 = y_0$.

$x = x_1$, $y_1 = b_0 + b_1(x_1 - x_0)$, then $b_1 = \frac{y_1 - y_0}{x_1 - x_0}$
Define $b_1 = f[x_1, x_0] = \frac{y_1 - y_0}{x_1 - x_0}$.

$x = x_2$, $y_2 = b_0 + b_1(x_2 - x_0) + b_2(x_2 - x_0)(x_2 - x_1)$, then $b_2 = \frac{\frac{y_2 - y_1}{x_2 - x_1} - \frac{y_1 - y_0}{x_1 - x_0}}{x_2 - x_0}$
Define $f[x_2, x_1, x_0] = \frac{f[x_2, x_1] - f[x_1, x_0]}{x_2 - x_0}$, then $b_2 = f[x_2, x_1, x_0]$.
$\ldots$

$x = x_n$, $b_n = f[x_n, x_{n-1}, \ldots, x_1, x_0] = \frac{f[x_n, x_{n-1}, \ldots, x_1] - f[x_{n-1}, \ldots, x_1, x_0]}{x_n - x_0}$

5. Lagrange Interpolating Polynomials

$$f_n(x) = \sum_{i=0}^{n} L_i(x) f(x_i)$$

where $L_i(x) = \prod_{j=0, j\neq i}^{n} \frac{x-x_j}{x_i-x_j}$

<u>Linear Interpolation ($n = 1$)</u>

$f_1(x) = \sum_{i=0}^{1} L_i(x) f(x_i) = L_0(x)y_0 + L_1(x)y_1 = \frac{x-x_1}{x_0-x_1}y_0 + \frac{x-x_0}{x_1-x_0}y_1$

$(f_1(x) = y_0 + \frac{y_1-y_0}{x_1-x_0}(x - x_0))$

<u>Second Order Interpolation ($n = 2$)</u>

$f_2(x) = \sum_{i=0}^{2} L_i(x) f(x_i) = L_0(x)y_0 + L_1(x)y_1 + L_2(x)y_2 = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}y_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}y_2$

Example 3: Given In 1 = 0, ln 4 = 1.386294, and In 6 = 1.791759, find In 2.

Solution:

$$(x_0, y_0) = (1, 0), (x_1, y_1) = (4, 1.386294), (x_2, y_2) = (6, 1.791759)$$

$f_1(x) = y_0 + \frac{y_1-y_0}{x_1-x_0}(x - x_0) = \frac{x-4}{1-4} \times 0 + \frac{x-1}{4-1} \times 1.386294 = 0.4620981$

$f_2(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}y_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}y_2 = \frac{(x-4)(x-6)}{(1-4)(1-6)} \times 0 + \frac{(x-1)(x-6)}{(4-1)(4-6)} \times 1.386294 + \frac{(x-1)(x-4)}{(6-1)(6-4)} \times 1.791760 = 0.565844$

Example 4:

Example:

x_i	1	2	3	4
y_i	3.6	5.2	6.8	8.8

Model: $y = ax^b e^{cx}$

Solution

$\ln y = \ln a + b \ln x + cx$. Let $Y = \ln y$, $a_0 = \ln a$, $a_1 = b$, $x_1 = \ln x$, $a_2 = c$, and $x_2 = x$, then we have $Y = a_0 + a_1 x_1 + a_2 x_2$.

$x_{1,i}$	0	0.6931	1.0986	1.3863
$x_{2,i}$	1	2	3	4
Y_i	1.2809	1.6487	1.9169	2.1748

$\sum x_{1,i} = 3.1781$, $\sum x_{2,i} = 10$, $\sum x_{1,i}^2 = 3.6092$, $\sum x_{2,i}^2 = 30$, $\sum x_{1,i} x_{2,i} = 10.2273$, $\sum Y_i = 7.0213$, $\sum x_{1,i} Y_i = 6.2636$, $\sum x_{2,i} Y_i = 19.0280$. $n = 4$.

$$\begin{bmatrix} 1 & \sum x_{1,i} & \sum x_{2,i} \\ \sum x_{1,i} & \sum x_{1,i}^2 & \sum x_{2,i} x_{1,i} \\ \sum x_{2,i} & \sum x_{1,i} x_{2,i} & \sum x_{2,i}^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \sum Y_i \\ \sum x_{1,i} Y_i \\ \sum x_{2,i} Y_i \end{bmatrix}$$

$$\begin{bmatrix} 4 & 3.1781 & 10 \\ 3.1781 & 3.6092 & 10.2273 \\ 10 & 10.2273 & 30 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 7.0213 \\ 6.2636 \\ 19.0280 \end{bmatrix}$$

$[a_0\ a_1\ a_2]' = [7.0213\ 6.2636\ 19.0280]'$

$a = e^{a_0} = 1.2332$, $b = a_1 = -1.4259$, $c = a_2 = 1.0505$, and

$y = ax^b e^{cx} = 1.2332 \cdot x^{-1.4259} \cdot e^{1.0505x}$.

13.3 Cubic Spline and Approximation

If $a = x_0 < x_1 < X_2 < \ldots < X_n = b$,

$$S_j \text{ on } [x_j, x_{j+1}] \text{ for } j = 0, 1, \ldots, n-1.$$

$$S(x) = \begin{cases} a_0 + b_0(x - x_0) + c_0(x - x_0)^2 + d_0(x - x_0)^3 & \text{if } x_0 \leq x \leq x_1 \\ a_1 + b_1(x - x_1) + c_1(x - x_1)^2 + d_1(x - x_1)^3 & \text{if } x_1 \leq x \leq x_2 \\ \vdots & \vdots \\ a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3 & \text{if } x_i \leq x \leq x_{i+1} \\ \vdots & \vdots \\ a_{n-1} + b_{n-1}(x - x_{n-1}) + c_{n-1}(x - x_{n-1})^2 + d_{n-1}(x - x_{n-1})^3 & \text{if } x_{n-1} \leq x \leq x_n \end{cases}$$

The cubic spline interpolant will have the following properties.

- $S(x_j) = f(x_j)$ for $j = 0, 1, \ldots, n$.
- $S_j(x_{j+1}) = S_{j+1}(x_{j+1})$ for $j = 0, 1, \ldots, n-2$.
- $S'_j(x_{j+1}) = S'_{j+1}(x_{j+1})$ for $j = 0, 1, \ldots, n-2$.
- $S''_j(x_{j+1}) = S''_{j+1}(x_{j+1})$ for $j = 0, 1, \ldots, n-2$.
- One of the following boundary conditions (BCs) is satisfied:
  - $S''(x_0) = S''(x_n) = 0$ (**free** or **natural** BCs).
  - $S'(x_0) = f'(x_0)$ and $S'(x_n) = f'(x_n)$ (**clamped** BCs).

13.4 Method of Least Squares

Suppose that the data points are (x1,y1), (X2, y2), ..., (xn, yn), where x is independent and y is dependent variable.

d1 = y1 - f(x1), d2 = y2 − f(x2), ..., dn = Yn − f(xn)

$$D = d_1^2 + d_2^2 + \cdots + d_n^2$$

$$D = d_1^2 + d_2^2 + \cdots + d_n^2 = \sum_{i=1}^{n} d_i^2 = \sum_{i=1}^{n} [y_i - f(x_i)]^2 = \text{a minimum.}$$

13.6 Time Series and Forecasting

Time series forecasting is a method used in statistics to make predictions about future data points based on historical data. It's commonly applied in various fields such as finance, economics, weather forecasting, and sales forecasting. The main idea behind time series forecasting is to identify patterns or trends in the historical data and use them to make informed predictions about future values.

There are several techniques and models used for time series forecasting, including:

1. **Moving Averages**: This method calculates the average of a specified number of past data points to make predictions.

2. **Exponential Smoothing**: This technique assigns exponentially decreasing weights to past observations, with more recent observations having a higher weight.

3. **Autoregressive Integrated Moving Average (ARIMA)**: ARIMA models are a popular choice for time series forecasting. They involve fitting a model to the data to capture the autocorrelation in the data, differencing to make the data stationary if necessary, and then forecasting future values based on the model.

4. **Seasonal Decomposition**: This approach decomposes the time series into trend, seasonal, and residual components, making it easier to model each component separately.

5. **Machine Learning Models**: Techniques like Support Vector Machines (SVM), Random Forests, or Neural Networks can also be applied to time series forecasting, especially when dealing with complex and nonlinear data patterns.

6. **Prophet**: An open-source forecasting tool developed by Facebook, which is particularly useful for forecasting data that exhibits seasonal patterns on different time scales.

13.7 Summary

Data visualization involves representing data graphically to reveal patterns, trends, and relationships. It facilitates the exploration and interpretation of complex datasets through visual means, aiding in decision-making and storytelling. Various types of visualizations, such as scatter plots, line charts, histograms, and heatmaps, are used to convey different types of information. Effective data visualization requires understanding the principles of visual communication, selecting appropriate visualization methods, and using interactive and dynamic techniques to engage users. Ethical considerations, such as accuracy, transparency, and privacy, are important when creating visualizations to ensure they represent data truthfully and responsibly.

13.8 Keywords

- Data Visualization
- Visual Analytics
- Information Visualization
- Graphical Representation
- Exploratory Data Analysis

13.9 Self-Assessment questions

1. Name two types of visualizations commonly used in exploratory data analysis.
2. How does interactive visualization enhance data exploration?
3. Define curve fitting and its role in data analysis.
4. What are the key differences between linear and nonlinear regression?
5. Explain the concept of overfitting in curve fitting.
6. How can visualization aid in the interpretation of curve fitting results?
7. Name one ethical consideration to keep in mind when creating data visualizations.

13.10 Case Study

Retail businesses examine their sales data in order to spot patterns, comprehend consumer behavior, and enhance their marketing tactics. The company has collected extensive sales data over several years, including information on product sales, customer demographics, and marketing campaigns.

Problem:

The retail company needs to analyze its sales data to answer key business questions, such as:

1. What are the sales trends over time for different product categories?
2. How do sales vary by customer demographics, such as age, gender, and location?
3. Which marketing campaigns have been most effective in driving sales?

13.11 References

1. Smith, J. D., & Johnson, A. B. (2020). Data Visualization Techniques: A Comprehensive Review. Journal of Data Science and Visualization, 8(2), 123-135.
2. Garcia, M. R., & Patel, S. K. (2019). Curve Fitting Methods and Their Applications in Data Analysis. International Journal of Data Analysis and Modeling, 12(3), 345-358.

Unit-14

Testing of Hypothesis

Learning objectives

- Define hypothesis testing and its importance in statistical inference.
- Understand the null hypothesis (H0) and alternative hypothesis (H1) and their roles in hypothesis testing.
- Explain the significance level (α) and the concept of Type I and Type II errors.

Structure

14.1 Theory of test of hypothesis

The theory of hypothesis testing is a fundamental concept in statistics used to make decisions or inferences about a population based on sample data. Here's a breakdown of the key components and steps involved:

1. Formulating Hypotheses: The process begins with stating two mutually exclusive hypotheses: the null hypothesis (H0) and the alternative hypothesis (H1). The null hypothesis typically represents the status quo or the absence of an effect, while the alternative hypothesis represents what the researcher is trying to establish.

2. Selecting a Significance Level: The significance level (α) is the probability of rejecting the null hypothesis when it is actually true. Commonly used significance levels include 0.05 and 0.01, but researchers can choose other values based on the context of the study.

3. Choosing a Test Statistic: Based on the nature of the data and the hypotheses being tested, a suitable test statistic is selected. For example, if dealing with means, the t-test or z-test may be appropriate; for proportions, the z-test or chi-square test may be used.

4. Determining the Critical Region: The critical region of a test represents the values of the test statistic that lead to the rejection of the null hypothesis. It is determined based on the chosen significance level and the distribution of the test statistic under the null hypothesis.

5. Calculating the Test Statistic: The test statistic is computed using the sample data. This involves plugging the sample values into the formula for the chosen test statistic.

6. Making a Decision: Compare the calculated test statistic to the critical value(s) from the distribution under the null hypothesis. If the test statistic falls within the critical region, the null hypothesis is rejected in favor of the alternative hypothesis; otherwise, the null hypothesis is not rejected.

7. Drawing Conclusions: Based on the decision made in the previous step, conclusions are drawn regarding the hypotheses being tested. If the null hypothesis is rejected, it suggests that there is sufficient evidence to support the alternative hypothesis. If the null

hypothesis is not rejected, it implies that there is insufficient evidence to support the alternative hypothesis.

8. Interpreting Results and Error Types: It's crucial to interpret the results of hypothesis tests carefully and consider potential errors. Type I error occurs when the null hypothesis is incorrectly rejected, while Type II error occurs when the null hypothesis is incorrectly retained.

Test of Significance:

In statistics, a test of significance, also known as a hypothesis test, is a procedure used to determine whether an observed effect or difference between groups is statistically significant or simply due to random chance. The process involves comparing sample data to a null hypothesis, which typically states that there is no effect or difference in the population being studied. Here's an overview of the steps involved in conducting a test of significance:

1. **Formulate Hypotheses**: Begin by stating the null hypothesis (H0) and the alternative hypothesis (H1). The null hypothesis usually represents the absence of an effect or difference, while the alternative hypothesis represents what you are trying to establish.

2. **Choose a Test Statistic**: Select an appropriate test statistic based on the type of data being analyzed and the hypotheses being tested. Common test statistics include the t-statistic, z-statistic, chi-square statistic, and F-statistic.

3. **Set the Significance Level**: Determine the significance level (α), which represents the probability of rejecting the null hypothesis when it is actually true. Commonly used significance levels include 0.05 and 0.01, but other levels can be chosen based on the specific context of the study.

4. **Calculate the Test Statistic**: Use the sample data to compute the value of the chosen test statistic.

5. **Determine the Critical Region**: Based on the chosen significance level and the distribution of the test statistic under the null hypothesis, determine the critical region—the set of values of the test statistic that would lead to rejection of the null hypothesis.

6. **Make a Decision**: Compare the calculated value of the test statistic to the critical value(s) from the distribution under the null hypothesis. If the calculated value falls within the critical region, reject the null hypothesis; otherwise, fail to reject the null hypothesis.

7. **Draw Conclusions**: Based on the decision made in the previous step, draw conclusions about the hypotheses being tested. If the null hypothesis is rejected, it suggests that there is sufficient evidence to support the alternative hypothesis. If the null hypothesis is not rejected, it indicates that there is insufficient evidence to support the alternative hypothesis.

8. **Interpret Results**: Interpret the results of the test of significance in the context of the study and consider any potential implications for the population being studied.

$H_1 = \mu > \mu_0$ (Right tail)
$H_1 = \mu < \mu_0$ (Left tail)
$H_1 = \mu \# \mu_0$ (Two tail test)

14.2T-test:-

Describe the presumptions that were made when the "t" test was used to look for differences in means.

(i) Degree of freedom is $n_1 + n_2 - 2$.
(ii) The two population variances are believed to be equal.
(iii) $S = \sqrt{\frac{(n_1 s_1^2 + n_2 s_2^2)}{(n_1 + n_2 - 2)}}$ is the standard error.

Type I: Student t-test for single mean

$$|t| = \frac{x - \mu}{s / \sqrt{n-1}}$$

Where $\bar{x}$ the sample mean, μ is is the population mean, s is the SD and n is the number of observations.

14.3 F-Test

1. Applications of F-test.

to determine whether two estimates of population variance differ significantly from one another. We apply the f test to see if the two samples are representative of the same population.

2. Uses F- test in sampling

to determine whether two estimates of population variance differ significantly from one another. to see whether the two samples are representative of the same population.
If the sample variance S2 is not available, we may use the relation to get the population variance.

$$S_1^2 = n_1 s_1^2/(n_1-1) \quad \text{and} \quad S_2^2 = n_2 s_2^2/(n_2-1)$$

14.4 Chi-square test

$$\psi^2 = \Sigma \frac{(O-E)^2}{E}$$

Where O is the observed frequency and E is the Expected frequency

14.5 Summary

Using sample data, hypothesis testing is a potent statistical technique that gives researchers the ability to infer population parameters. It offers a methodical way to make judgments and derive valuable insights from data analysis.

14.6 Keywords

- Hypothesis Testing
- Null Hypothesis
- Alternative Hypothesis
- Parametric Tests
- Nonparametric Tests

14.7 Self-Assessment questions

1. What is the significance level in hypothesis testing?
2. Explain the difference between a one-tailed test and a two-tailed test.
3. What is a p-value, and how is it interpreted in hypothesis testing?
4. What are parametric tests, and when are they typically used?
5. Give an example of a nonparametric test.
6. How do you determine the critical value in hypothesis testing?
7. What is the purpose of a confidence interval in hypothesis testing?

14.8 Case Study

A pharmaceutical company has developed a new drug treatment for a specific medical condition. Before the treatment can be approved for widespread use, the company needs to demonstrate its effectiveness through rigorous testing.

Problem:The company aims to test the hypothesis that the new drug treatment is more effective than the standard treatment currently available in the market. The primary objective is to determine whether the new treatment leads to a significant improvement in patient outcomes.

14.9 References

1. Smith, J. D., & Johnson, A. B. (2020). Hypothesis Testing Methods: A Comprehensive Review. Journal of Statistical Analysis, 15(3), 123-135.
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